Einstein-Schrödinger theory in the presence of zero-point fluctuations

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Abstract. The Einstein-Schrödinger theory is modified by adding a cosmological constant contribution caused by zero-point fluctuations. This cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger's "bare" cosmological constant which multiplies the nonsymmetric fundamental tensor, such that the total "physical" cosmological constant matches measurement. We first derive the field equations of the theory from a Lagrangian density. We show that the divergence of the Einstein equations vanishes using the Christoffel connection formed from the symmetric metric, allowing additional fields to be included in the same manner as with ordinary general relativity. We show that the field equations match the ordinary electro-vac Einstein and Maxwell equations except for additional terms which are $< 10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. We also show that the theory avoids ghosts in an unusual way. We show that the Einstein-Infeld-Hoffmann (EIH) equations of motion for this theory match the equations of motion for Einstein-Maxwell theory to Newtonian/Coulombian order, which proves the existence of a Lorentz force. We derive an exact electric monopole solution, and show that it matches the Reissner-Nordström solution except for additional terms which are $\sim 10^{-66}$ of the usual terms for worst-case radii accessible to measurement. Finally, we show that the theory becomes exactly electro-vac Einstein-Maxwell theory in the limit as the cosmological constant from zero-point fluctuations goes to infinity.

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1. Introduction

The Einstein-Schrödinger theory without a cosmological constant was originally proposed by Einstein and Straus[1, 2, 3, 4, 5]. Schrödinger generalized the theory to include a cosmological constant, and showed that the theory can be derived from a very simple Lagrangian density if this cosmological constant is assumed to be non-zero[6, 7, 8]. This more general theory is usually called Schrödinger's Affine Field Theory or the Einstein-Schrödinger Theory. The theory is a generalization of ordinary general relativity which allows a non-symmetric fundamental tensor and connection. Einstein and Schrödinger suspected that the fundamental tensor might contain both the metric and the electromagnetic field, but this was never demonstrated.

In this paper we show that a well motivated modification of the Einstein-Schrödinger theory does indeed closely approximate ordinary electro-vac Einstein-Maxwell theory, the modification being the addition of a cosmological constant caused by zero-point fluctuations[10, 11, 12]. This cosmological constant which

multiplies the symmetric metric $g_{\nu\mu}$ is assumed to be nearly cancelled by Schrödinger's "bare" cosmological constant which multiplies the nonsymmetric fundamental tensor $N_{\nu\mu}$, resulting in a total "physical" cosmological constant which is consistent with measurement. This is essentially the same as vacuum energy renormalization in quantum field theory, and it can be viewed as a kind of zeroth order quantization effect. Note that we are not attempting to quantize the Einstein-Schrödinger theory. We are considering the classical Einstein-Schrödinger theory, but with a modification to account for a quantum mechanical effect.

For electro-vac Einstein-Maxwell theory, the Einstein-Infeld-Hoffmann (EIH) method[23, 24] allows the equations-of-motion for both charged and neutral particles to be derived directly from the field equations. When the EIH method was originally applied to the Einstein-Schrödinger theory, no Lorentz force was found between charged particles[25, 26]. This is the primary reason that the Einstein-Schrödinger theory was abandoned by most researchers long ago. It is significant that no cosmological constant was assumed in [25, 26]. Quantum field theory predicts that zero-point fluctuations should cause a very large cosmological constant. Also, recent evidence suggests an ordinary cosmological constant as the most likely reason that the expansion of the universe is accelerating[28, 29]. It is shown in §6 that when a cosmological constant is assumed, and when $g_{\mu\nu}$ and $F_{\mu\nu}$ are defined differently than in [25, 26], the EIH method definitely predicts a Lorentz force. This can be confirmed by including charged matter terms in the Lagrangian density as in [22], in which case the exact Lorentz-force equation can be derived from the theory.

In much previous work on the Einstein-Schrödinger theory, the electromagnetic field is assumed to be the dual of $N_{[\nu\mu]}$ or $N^{*[\alpha\rho]} = \varepsilon^{\alpha\rho\nu\mu}N_{[\nu\mu]}/2$, as originally proposed by Einstein[5]. However, all efforts have failed to connect the resulting current source $(\varepsilon^{\rho\alpha\nu\mu}N_{[\nu\mu]})_{;\alpha}/2$ to a real charge current. With this definition, a Lorentz-like force can be demonstrated[14, 17, 18] without a cosmological constant. However, the solutions[19, 20] that must be used for test particles have bad asymptotic behavior, such as a radial electrostatic field which is independent of radius at large distances. Our electric monopole solution in §7 has no such problem.

Recent work[31, 32, 33] shows that the original Einstein-Schrödinger theory has problems with negative energy "ghosts". As will be seen in §5, this problem is avoided in the present theory in an unusual way. In most of the work referenced above, the electromagnetic field is assumed to be an "added on" field, unrelated to $N_{[\nu\mu]}$. Because this approach is not taken in the present theory, problems[34, 35, 36] caused by the coupling of $N_{\nu\mu}$ to the electromagnetic field do not apply here.

In some previous work, the equations of the Einstein-Schrödinger theory are significantly modified[37, 38]. In some theories[39, 40, 41, 42], $N_{[\tau\rho]}$ is interpreted as the electromagnetic field, and a Lorentz force is derived, but only because a term $\sqrt{-N}N^{\dashv[\rho\tau]}N_{[\tau\rho]}$ is appended onto the Einstein-Schrödinger Lagrangian density. None of these theories have been shown to approximate Einstein-Maxwell theory close enough to agree with experiment, and the modifications assumed in these theories have no clear physical motivation. Neither criticism applies to the present theory.

Recently there has been much interest in Born-Infeld electrodynamics[43, 44] because it appears to result from string theory. The similarity of the Einstein-Schrödinger theory to Born-Infeld electrodynamics is noted by [45] who suggest a connection to string theory. While a possible connection between the Einstein-Schrödinger theory and string theory is beyond the scope of this paper, it is nevertheless an additional reason to investigate the theory.

This paper is organized as follows. In §2-3 we first derive the field equations of this modified Einstein-Schrödinger theory from a Lagrangian density. We show that the divergence of the Einstein equations vanishes using the Christoffel connection formed from the symmetric metric, allowing additional (non-electromagnetic) fields to be included in the same manner as with ordinary general relativity. In §4-5 we show that the field equations match the ordinary electro-vac Einstein and Maxwell equations except for additional terms which are $< 10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. We also show that the theory avoids ghosts in an unusual way. In §6 we derive the Einstein-Infeld-Hoffmann (EIH) equations of motion for this theory and show that they match the equations of motion for Einstein-Maxwell theory to Newtonian/Coulombian order, which proves the existence of a Lorentz force. In §7 derive an exact electric monopole solution, and show that it matches the Reissner-Nordström solution except for additional terms which are $\sim 10^{-66}$ of the usual terms for worst-case radii accessible to measurement.

2. The Lagrangian Density

Ordinary vacuum general relativity can be derived from a Palatini Lagrangian density,

$$\mathcal{L}(\Gamma^\lambda_{\rho\tau}, g_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-g}\,[\,g^{\mu\nu}R_{\nu\mu}(\Gamma) + (n-2)\Lambda_b\,]\,. \tag{1}$$

Here and throughout this paper we are assuming that $n = 4$, but the dimension "n" will be included in the equations to show how easily the results can be generalized to arbitrary dimension. The original unmodified Einstein-Schrödinger theory[6, 7, 8, 1, 2, 3, 4, 5] can be derived from a generalization of (1) formed from a connection $\widehat{\Gamma}^\alpha_{\nu\mu}$ and a fundamental tensor $N_{\nu\mu}$ with no symmetry properties (see Appendix E for an alternative derivation),

$$\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) + (n-2)\Lambda_b\right]. \tag{2}$$

Our theory includes a cosmological constant Λ_z caused by zero-point fluctuations,

$$\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) = -\frac{1}{16\pi}\left[\sqrt{-N}N^{\dashv\mu\nu}\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) + (n-2)\Lambda_b\sqrt{-N} + (n-2)\Lambda_z\sqrt{-g}\,\right], \tag{3}$$

where the "bare" Λ_b obeys $\Lambda_b \approx -\Lambda_z$ so that the "physical" Λ matches measurement,

$$\Lambda = \Lambda_b + \Lambda_z, \tag{4}$$

and the metric and electromagnetic potential are defined as

$$\sqrt{-g}\,g^{\mu\nu} = \sqrt{-N}N^{\dashv(\mu\nu)}, \tag{5}$$

$$A_\nu = \frac{1}{2(n-1)}\,\widehat{\Gamma}^\sigma_{[\sigma\nu]}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{6}$$

Here and throughout this paper we use geometrized units with $c=G=1$, the symbols ( ) and [ ] around indices indicate symmetrization and antisymmetrization, "n" is the dimension, $N = det(N_{\mu\nu})$, and $N^{\dashv\sigma\nu}$ is the inverse of $N_{\nu\mu}$ so that $N^{\dashv\sigma\nu}N_{\nu\mu} = \delta^\sigma_\mu$. In (3), $\mathcal{R}_{\nu\mu}(\widehat{\Gamma})$ is a form of the so-called Hermitianized Ricci tensor[1],

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \widehat{\Gamma}^\alpha_{\nu\mu,\alpha} - \widehat{\Gamma}^\alpha_{(\alpha(\nu),\mu)} + \widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{(\alpha\sigma)} - \widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu} - \widehat{\Gamma}^\tau_{[\tau\nu]}\widehat{\Gamma}^\alpha_{[\alpha\mu]}/(n-1). \tag{7}$$

This tensor reduces to the ordinary Ricci tensor for symmetric fields, where we have $\Gamma^\alpha_{[\nu\mu]} = 0$ and $\Gamma^\alpha_{\alpha[\nu,\mu]} = R^\alpha{}_{\alpha\mu\nu}/2 = 0$.

It is convenient to decompose $\widehat{\Gamma}^{\alpha}_{\nu\mu}$ into another connection $\tilde{\Gamma}^{\alpha}_{\nu\mu}$, and A_σ from (6),

$$\widehat{\Gamma}^{\alpha}_{\nu\mu} = \tilde{\Gamma}^{\alpha}_{\nu\mu} + (\delta^{\alpha}_{\mu} A_\nu - \delta^{\alpha}_{\nu} A_\mu)\sqrt{2}\, i\Lambda_b^{1/2}, \tag{8}$$

$$\text{where}\quad \tilde{\Gamma}^{\alpha}_{\nu\mu} = \widehat{\Gamma}^{\alpha}_{\nu\mu} + (\delta^{\alpha}_{\mu}\widehat{\Gamma}^{\sigma}_{[\sigma\nu]} - \delta^{\alpha}_{\nu}\widehat{\Gamma}^{\sigma}_{[\sigma\mu]})/(n-1). \tag{9}$$

By contracting (9) on the right and left we see that $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ has the symmetry

$$\tilde{\Gamma}^{\alpha}_{\nu\alpha} = \widehat{\Gamma}^{\alpha}_{(\nu\alpha)} = \tilde{\Gamma}^{\alpha}_{\alpha\nu}, \tag{10}$$

so it has only n^3–n independent components. Using $\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + 2A_{[\nu,\mu]}\sqrt{2}\, i\Lambda_b^{1/2}$ from (A.8), the Lagrangian density (3) can be rewritten in terms of $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ and A_σ,

$$\mathcal{L} = -\frac{1}{16\pi}\left[\sqrt{-N}N^{\dashv\mu\nu}(\tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{2}\, i\Lambda_b^{1/2}) + (n-2)\Lambda_b\sqrt{-N} + (n-2)\Lambda_z\sqrt{-g}\right]. \tag{11}$$

Here $\tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\tilde{\Gamma})$, and from (10) the Hermitianized Ricci tensor (7) simplifies to

$$\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\Gamma}^{\alpha}_{\nu\mu,\alpha} - \tilde{\Gamma}^{\alpha}_{\alpha(\nu,\mu)} + \tilde{\Gamma}^{\sigma}_{\nu\mu}\tilde{\Gamma}^{\alpha}_{\sigma\alpha} - \tilde{\Gamma}^{\sigma}_{\nu\alpha}\tilde{\Gamma}^{\alpha}_{\sigma\mu}. \tag{12}$$

From (8,10), $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ and A_ν fully parameterize $\widehat{\Gamma}^{\alpha}_{\nu\mu}$ and can be treated as independent variables. So when we set $\delta\mathcal{L}/\delta\tilde{\Gamma}^{\alpha}_{\nu\mu} = 0$ and $\delta\mathcal{L}/\delta A_\nu = 0$, the same field equations must result as with $\delta\mathcal{L}/\delta\widehat{\Gamma}^{\alpha}_{\nu\mu} = 0$. It is simpler to calculate the field equations using $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ and A_ν instead of $\widehat{\Gamma}^{\alpha}_{\nu\mu}$, so we will follow this method.

We will usually assume that Λ_z is limited by a cutoff frequency[46, 47, 48, 50]

$$\omega_c \sim 1/l_P, \tag{13}$$

where $l_P = (\text{Planck length}) = \sqrt{\hbar G/c^3} = 1.6\times 10^{-33}cm$. Then from (4,13) and assuming all of the known fundamental particles we have[10],

$$\Lambda_b \approx -\Lambda_z \sim C_z\omega_c^4 l_P^2 \sim 10^{66}\,\text{cm}^{-2}, \tag{14}$$

$$C_z = \frac{1}{2\pi}\left(\begin{matrix}\text{fermion}\\ \text{spin states}\end{matrix} - \begin{matrix}\text{boson}\\ \text{spin states}\end{matrix}\right) \sim \frac{60}{2\pi} \tag{15}$$

and from astronomical measurements[27, 28, 29, 30]

$$\Lambda \approx 1.4\times 10^{-56}\text{cm}^{-2}, \qquad \Lambda/\Lambda_b \sim 10^{-122}. \tag{16}$$

However, it might be more correct to fully renormalize with $\omega_c\to\infty$, $|\Lambda_z|\to\infty$, $\Lambda_b\to\infty$ as in quantum electrodynamics. To account for this possibility we will prove that

$$\lim_{\Lambda_b\to\infty}\left(\begin{matrix}\Lambda\text{-renormalized}\\ \text{Einstein-Schrödinger theory}\end{matrix}\right) = \left(\begin{matrix}\text{Einstein-Maxwell}\\ \text{theory}\end{matrix}\right). \tag{17}$$

The Hermitianized Ricci tensor (7) has the following invariance properties

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^T) = \mathcal{R}_{\mu\nu}(\widehat{\Gamma}), \qquad\qquad (\text{T} = \text{transpose}) \tag{18}$$

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^{\alpha}_{\rho\tau} + \delta^{\alpha}_{[\rho}\varphi_{,\tau]}) = \mathcal{R}_{\nu\mu}(\widehat{\Gamma}^{\alpha}_{\rho\tau}) \quad \text{for an arbitrary } \varphi(x^\sigma). \tag{19}$$

From (18,19), the Lagrangian densities (3,11) are invariant under charge conjugation,

$$Q\to -Q,\;\; A_\sigma\to -A_\sigma,\;\; \tilde{\Gamma}^{\alpha}_{\nu\mu}\to\tilde{\Gamma}^{\alpha}_{\mu\nu},\;\; \widehat{\Gamma}^{\alpha}_{\nu\mu}\to\widehat{\Gamma}^{\alpha}_{\mu\nu},\;\; N_{\nu\mu}\to N_{\mu\nu},\;\; N^{\dashv\nu\mu}\to N^{\dashv\mu\nu}, \tag{20}$$

and also under an electromagnetic gauge transformation

$$\psi\to\psi e^{i\phi},\;\; A_\alpha\to A_\alpha - \frac{\hbar}{Q}\phi_{,\alpha},\;\; \tilde{\Gamma}^{\alpha}_{\rho\tau}\to\tilde{\Gamma}^{\alpha}_{\rho\tau},\;\; \widehat{\Gamma}^{\alpha}_{\rho\tau}\to\widehat{\Gamma}^{\alpha}_{\rho\tau} + \frac{2\hbar}{Q}\delta^{\alpha}_{[\rho}\phi_{,\tau]}\sqrt{2}\, i\Lambda_b^{1/2}. \tag{21}$$

If $\Lambda_b > 0$, $\Lambda_z < 0$ as in (14,15) then $\tilde{\Gamma}^{\alpha}_{\nu\mu}$, $\widehat{\Gamma}^{\alpha}_{\nu\mu}$, $N_{\nu\mu}$ and $N^{\dashv\nu\mu}$ are all Hermitian, $\tilde{\mathcal{R}}_{\nu\mu}$ and $\mathcal{R}_{\nu\mu}(\widehat{\Gamma})$ are Hermitian from (18), and $g_{\nu\mu}$, A_σ and $\mathcal{L}$ are real from (5,6,3,11). If instead $\Lambda_b < 0$, $\Lambda_z > 0$, then all of the fields are real.

Note that (5) defines $g^{\mu\nu}$ unambiguously because $\sqrt{-g}=[-det(\sqrt{-g}\,g^{\mu\nu})]^{1/(n-2)}$. In this theory the metric $g_{\mu\nu}$ is used for measuring space-time intervals, and for calculating geodesics, and for raising and lowering of indices. The covariant derivative ";" is always done using the Christoffel connection formed from $g_{\mu\nu}$,

$$\Gamma^\alpha_{\nu\mu} = \frac{1}{2}\,g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\nu\mu,\sigma}). \tag{22}$$

With the metric (5), the divergence of the Einstein equations vanishes when using (22) for the covariant derivative. And when $N_{\mu\nu}$ and $\widehat{\Gamma}^\alpha_{\mu\nu}$ are symmetric, the definition (5) requires $g_{\mu\nu} = N_{\mu\nu}$, the definition (6) requires $A_\sigma = 0$, and the theory reduces to ordinary general relativity without electromagnetism.

The electromagnetic field is defined in terms of the potential (6) as usual

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{23}$$

However, we will also define a lowercase $f_{\mu\nu}$

$$\sqrt{-g}\,f^{\mu\nu} = \sqrt{-N}N^{\dashv[\nu\mu]}\Lambda_b^{1/2}/\sqrt{2}\,i. \tag{24}$$

Then from (5), $g^{\mu\nu}$ and $f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ are the symmetric and antisymmetric parts of a total field,

$$W^{\mu\nu} = (\sqrt{-N}/\sqrt{-g}\,)N^{\dashv\nu\mu} = g^{\mu\nu} + f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{25}$$

We will see that the field equations require $f_{\mu\nu}\approx F_{\mu\nu}$ to a very high precision, so it is mainly just a matter of terminology which one is called the electromagnetic field.

Note there are many possible nonsymmetric generalizations of the Ricci tensor besides the Hermitianized Ricci tensor $\mathcal{R}_{\nu\mu}(\widehat{\Gamma})$ from (7) and the ordinary Ricci tensor $R_{\nu\mu}(\widehat{\Gamma})$. For example, we could form any weighted average of $R_{\nu\mu}(\widehat{\Gamma})$, $R_{\mu\nu}(\widehat{\Gamma})$, $R_{\nu\mu}(\widehat{\Gamma}^T)$ and $R_{\mu\nu}(\widehat{\Gamma}^T)$, and then add any linear combination of the tensors $R^\alpha{}_{\alpha\nu\mu}(\widehat{\Gamma})$, $R^\alpha{}_{\alpha\nu\mu}(\widehat{\Gamma}^T)$, $\widehat{\Gamma}^\alpha_{[\nu\mu]}\widehat{\Gamma}^\sigma_{[\sigma\alpha]}$, $\widehat{\Gamma}^\alpha_{[\nu\sigma]}\widehat{\Gamma}^\sigma_{[\mu\alpha]}$, and $\widehat{\Gamma}^\alpha_{[\alpha\nu]}\widehat{\Gamma}^\sigma_{[\sigma\mu]}$. All of these generalized Ricci tensors would be linear in $\widehat{\Gamma}^\alpha_{\nu\mu,\sigma}$, quadratic in $\widehat{\Gamma}^\alpha_{\nu\mu}$, and would reduce to the ordinary Ricci tensor for symmetric fields. Even if we limit the tensor to only four terms, there are still eight possibilities. We assert that invariance properties like (18,19) are the most sensible way to choose among the different alternatives, not criteria such as the number of terms in the expression.

To include additional fields, we would simply append a matter term $\mathcal{L}_m$ onto (11), and this term is assumed to be formed with the metric $g_{\mu\nu}$, not with $N_{\mu\nu}$. This $\mathcal{L}_m$ may also contain the vector A_μ from (6), and charged spin-0 or spin-1/2 fields as in [22]. An $\mathcal{L}_m$ containing all of the additional fields of the Standard Model could even be included. Of course we would not want to add a $\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu}$ term because this term is effectively already contained in the theory. In this paper we will only be considering the Lagrangian density (11) with no $\mathcal{L}_m$ term appended onto it, which is our equivalent of electro-vac Einstein-Maxwell theory.

Finally, let us discuss some notation issues. We use the symbol $\Gamma^\alpha_{\nu\mu}$ for the Christoffel connection (22) whereas Einstein used it for our $\tilde{\Gamma}^\alpha_{\nu\mu}$ and Schrödinger used it for our $\widehat{\Gamma}^\alpha_{\nu\mu}$. We use the symbol $g_{\mu\nu}$ for the symmetric metric (5) whereas Einstein and Schrödinger both used it for our $N_{\mu\nu}$, the nonsymmetric fundamental tensor. Also, to represent the inverse of $N_{\alpha\mu}$ we use $N^{\dashv\sigma\alpha}$ instead of the more conventional $N^{\alpha\sigma}$, because this latter notation would be ambiguous when using $g^{\mu\nu}$ to raise indices. While our notation differs from previous literature on the Einstein-Schrödinger theory, this change is required by our explicit metric definition, and it is necessary to be consistent with the much larger body of literature on Einstein-Maxwell theory.

3. Derivation of the Field Equations

Here we will derive the field equations resulting from the Lagrangian density (11). Setting $\delta\mathcal{L}/\delta\widehat{\Gamma}^\alpha_{\nu\mu}\!=\!0$ will give the connection equations, and we will see that Ampere's law can be derived from these. However, as discussed previously, the same field equations must result if we instead use $\tilde\Gamma^\alpha_{\nu\mu}$ and A_τ as the independent variables, and since this is simpler we will follow this method. Setting $\delta\mathcal{L}/\delta A_\tau=0$ and using (24) gives Ampere's law,

$$0=\frac{4\pi}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial A_\tau}-\left(\frac{\partial\mathcal{L}}{\partial A_{\tau,\omega}}\right)_{\!,\,\omega}\right] \tag{26}$$

$$=\frac{\sqrt{2}\,i\Lambda_b^{1/2}}{2\sqrt{-g}}(\sqrt{-N}N^{\dashv[\omega\tau]})_{,\,\omega}=\frac{1}{\sqrt{-g}}(\sqrt{-g}f^{\omega\tau})_{,\,\omega}=f^{\omega\tau}{}_{;\,\omega}. \tag{27}$$

To calculate $\delta\mathcal{L}/\delta\tilde\Gamma^\alpha_{\nu\mu}$ let us first define

$$\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}=\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho}}-\left(\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho,\omega}}\right)_{\!,\,\omega}\;.... \tag{28}$$

Then from (11,12,27) we can calculate,

$$\begin{aligned}-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}&=2\sqrt{-N}N^{\dashv\mu\nu}(\delta^\sigma_\beta\delta^\tau_\nu\delta^\rho_{[\mu|}\tilde\Gamma^\alpha_{\sigma|\alpha]}+\tilde\Gamma^\sigma_{\nu[\mu|}\delta^\alpha_\beta\delta^\tau_\sigma\delta^\rho_{|\alpha]})\\&\quad-2(\sqrt{-N}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\nu\delta^\rho_{[\mu}\delta^\omega_{\alpha]})_{,\,\omega}-(\sqrt{-N}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\alpha\delta^\rho_{[\nu}\delta^\omega_{\mu]})_{,\omega}\\&=-(\sqrt{-N}N^{\dashv\rho\tau})_{,\,\beta}-\tilde\Gamma^\rho_{\beta\mu}\sqrt{-N}N^{\dashv\mu\tau}-\tilde\Gamma^\tau_{\nu\beta}\sqrt{-N}N^{\dashv\rho\nu}+\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}\\&\quad+\delta^\rho_\beta((\sqrt{-N}N^{\dashv\,\omega\tau})_{,\,\omega}+\tilde\Gamma^\tau_{\nu\mu}\sqrt{-N}N^{\dashv\mu\nu}),\end{aligned} \tag{29}$$

$$-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}}=2(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\,\omega}=0, \tag{30}$$

$$-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\tau\alpha}}=(n\!-\!1)((\sqrt{-N}N^{\dashv\,\omega\tau})_{,\,\omega}+\tilde\Gamma^\tau_{\nu\mu}\sqrt{-N}N^{\dashv\mu\nu}). \tag{31}$$

In these last two equations, the index contractions occur after the derivatives. At this point we must be careful. Because $\tilde\Gamma^\alpha_{\nu\mu}$ has the symmetry (10), it has only n^3-n independent components, so there can only be n^3-n independent field equations associated with it. It is shown in Appendix C that instead of just setting (29) to zero, the field equations associated with such a field are given by the expression,

$$0=16\pi\left[\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}-\frac{\delta^\tau_\beta}{(n\!-\!1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}}-\frac{\delta^\rho_\beta}{(n\!-\!1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\tau\alpha}}\right] \tag{32}$$

$$=(\sqrt{-N}N^{\dashv\rho\tau})_{,\,\beta}+\tilde\Gamma^\tau_{\sigma\beta}\sqrt{-N}N^{\dashv\rho\sigma}+\tilde\Gamma^\rho_{\beta\sigma}\sqrt{-N}N^{\dashv\sigma\tau}-\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}. \tag{33}$$

These are the connection equations, like $(\sqrt{-g}g^{\rho\tau})_{;\beta}\!=\!0$ in the symmetric case.

From the definition of matrix inverse $N^{\dashv\rho\tau}\!=(1/N)\partial N/\partial N_{\tau\rho}$, $N^{\dashv\rho\tau}N_{\tau\mu}\!=\delta^\rho_\mu$ we get the identity

$$(\sqrt{-N}\,)_{,\beta}=\frac{\partial\sqrt{-N}}{\partial N_{\tau\rho}}N_{\tau\rho,\beta}=\frac{\sqrt{-N}}{2}N^{\dashv\rho\tau}N_{\tau\rho,\beta}=-\frac{\sqrt{-N}}{2}N^{\dashv\rho\tau}{}_{,\beta}N_{\tau\rho}. \tag{34}$$

Contracting (33) with $N_{\tau\rho}$ using (34,10) and dividing by $(n\!-\!2)$ gives

$$(\sqrt{-N}\,)_{,\,\beta}-\tilde\Gamma^\alpha_{\alpha\beta}\sqrt{-N}=0. \tag{35}$$

This shows that the tensor $\tilde\Gamma^\alpha_{\alpha[\nu,\mu]}=\tilde R^\alpha{}_{\alpha\mu\nu}/2$ vanishes,

$$\tilde\Gamma^\alpha_{\alpha[\nu,\mu]} = (ln\sqrt{-N}\,)_{,[\nu,\mu]} = 0. \tag{36}$$

From (33,35) we get the contravariant connection equations,

$$N^{\dashv\rho\tau}{}_{,\beta}+\tilde\Gamma^\tau_{\sigma\beta}N^{\dashv\rho\sigma}+\tilde\Gamma^\rho_{\beta\sigma}N^{\dashv\sigma\tau} = 0. \tag{37}$$

Multiplying this by $-N_{\nu\rho}N_{\tau\mu}$ gives the covariant connection equations,

$$N_{\nu\mu,\beta}-\tilde\Gamma^\alpha_{\nu\beta}N_{\alpha\mu}-\tilde\Gamma^\alpha_{\beta\mu}N_{\nu\alpha} = 0. \tag{38}$$

Setting $\delta\mathcal{L}/\delta N_{\nu\mu}=0$ will give the Einstein equations. However, the same field equations must result if we instead use $\sqrt{-N}N^{\dashv\mu\nu}$ as the independent variable, and since this is simpler we will follow this method. Before calculating the field equations, we need some preliminary results. From (5) we get,

$$\frac{\partial\,(\sqrt{-g}g^{\rho\tau})}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \delta^{(\rho}_\mu\delta^{\tau)}_\nu, \tag{39}$$

$$\frac{\partial(g_{\tau\sigma}/\sqrt{-g})}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = -\,\frac{g_{\tau(\nu}g_{\mu)\sigma}}{\sqrt{-g}\sqrt{-g}} \qquad \left(\text{because }\ \frac{\partial\,(\sqrt{-g}g^{\rho\tau}g_{\tau\sigma}/\sqrt{-g})}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = 0\right). \tag{40}$$

Using (5) and the identities $det(sM)=s^n det(M)$, $det(M^{-1})=1/det(M)$ gives

$$\sqrt{-N} = (-det(\sqrt{-N}N^{\dashv\cdot\cdot}))^{1/(n-2)}, \tag{41}$$

$$\sqrt{-g}\, = (-det(\sqrt{-g}\,g^{\cdot\cdot}))^{1/(n-2)} = (-det(\sqrt{-N}N^{\dashv(\cdot\cdot)}))^{1/(n-2)}. \tag{42}$$

Using (41,42) and the identity $\partial(det(M^{\cdot\cdot}))/\partial M^{\mu\nu} = M^{-1}_{\nu\mu}det(M^{\cdot\cdot})$ gives

$$\frac{\partial\sqrt{-N}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \frac{(-det(\sqrt{-N}N^{\dashv\cdot\cdot}))^{1/(n-2)-1+1}}{(n-2)}\frac{N_{\nu\mu}}{\sqrt{-N}} = \frac{N_{\nu\mu}}{(n-2)}, \tag{43}$$

$$\frac{\partial\sqrt{-g}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \frac{(-det(\sqrt{-g}\,g^{\cdot\cdot}))^{1/(n-2)-1+1}}{(n-2)}\frac{g_{\nu\mu}}{\sqrt{-g}} = \frac{g_{\nu\mu}}{(n-2)}. \tag{44}$$

Note that from (39,40,44), if there was a matter term $\mathcal{L}_m$ in (11) which depended only on $g_{\nu\mu}$ and not on $N_{\nu\mu}$, then $\partial\mathcal{L}_m/\partial(\sqrt{-N}N^{\dashv\mu\nu}) = \partial\mathcal{L}_m/\partial(\sqrt{-g}\,g^{\mu\nu})$. Setting $\delta\mathcal{L}/\delta(\sqrt{-N}N^{\dashv\mu\nu})=0$ and using (43,44) gives,

$$0 = \,-\,16\pi\left[\frac{\partial\mathcal{L}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} - \left(\frac{\partial\mathcal{L}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})_{,\,\omega}}\right)_{,\,\omega}\right] \tag{45}$$

$$= \tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{2}\,i\Lambda_b^{1/2} + \Lambda_b N_{\nu\mu} + \Lambda_z g_{\nu\mu}. \tag{46}$$

Using the definition (23), the antisymmetric part of this is

$$N_{[\nu\mu]} = F_{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2} - \tilde{\mathcal{R}}_{[\nu\mu]}\Lambda_b^{-1}. \tag{47}$$

Taking the symmetric part of (46) and the curl of (47) and repeating (38,10) gives the field equations in the form usually used to define the Einstein-Schrödinger theory,

$$\tilde{\mathcal{R}}_{(\nu\mu)} + \Lambda_b N_{(\nu\mu)} + \Lambda_z g_{\nu\mu} = 0, \tag{48}$$

$$\tilde{\mathcal{R}}_{[\nu\mu,\sigma]} + \Lambda_b N_{[\nu\mu,\sigma]} = 0, \tag{49}$$

$$N_{\nu\mu,\beta}-\tilde\Gamma^\alpha_{\nu\beta}N_{\alpha\mu}-\tilde\Gamma^\alpha_{\beta\mu}N_{\nu\alpha} = 0, \tag{50}$$

$$\tilde\Gamma^\alpha_{\beta\alpha} = \tilde\Gamma^\alpha_{\alpha\beta}. \tag{51}$$

If desired we could start from these equations instead of the Lagrangian density (3). That is, the symmetric part of (46) comes from (48), and the antisymmetric part of (46)

is implied by (49) for some A_μ. Also, Ampere's law (27) can be derived from (50,51) by employing (37,34) to get (35,33), and then antisymmetrizing and contracting (33).

The Einstein equations are obtained by combining (48) with its contraction,

$$\tilde{G}_{\nu\mu} + \Lambda_b\left(N_{(\nu\mu)} - \frac{1}{2}\,g_{\nu\mu}N^\rho_\rho\right) + \Lambda_z\left(1 - \frac{n}{2}\right)g_{\nu\mu} = 0. \tag{52}$$

where we define

$$\tilde{G}_{\nu\mu} = \tilde{\mathcal{R}}_{(\nu\mu)} - \frac{1}{2}g_{\nu\mu}\tilde{\mathcal{R}}^\rho_\rho. \tag{53}$$

A generalized contracted Bianchi identity for this theory is derived in [3] using only the connection equations (50) and the symmetry (51). When expressed in terms of our metric (5) and the definitions (24,53), this identity becomes[15, 16, 22]

$$\tilde{G}^\sigma_{\nu;\,\sigma} = \frac{3}{2}f^{\sigma\rho}\tilde{\mathcal{R}}_{[\sigma\rho,\nu]}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{54}$$

Another useful identity is derived in Appendix B using only the definitions (5,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$,

$$\left(N^{(\mu}{}_{\nu)} - \frac{1}{2}\delta^\mu_\nu N^\rho_\rho\right)_{;\,\mu} = \left(\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho,\nu]} + f^{\sigma\rho}{}_{;\sigma}N_{[\rho\nu]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{55}$$

Using (54,55,27,49) we see that the divergence of the Einstein equations (52) vanishes

$$\left[\tilde{G}^\mu_\nu + \Lambda_b\left(N^{(\mu}{}_{\nu)} - \frac{1}{2}\delta^\mu_\nu N^\rho_\rho\right) + \Lambda_z\left(1 - \frac{n}{2}\right)\delta^\mu_\nu\right]_{;\,\mu}$$
$$= \frac{3}{2}f^{\sigma\rho}\tilde{\mathcal{R}}_{[\sigma\rho,\nu]}\sqrt{2}\,i\Lambda_b^{-1/2} + \Lambda_b\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho,\nu]}\sqrt{2}\,i\Lambda_b^{-1/2} = 0. \tag{56}$$

This is why the metric (5) was assumed, because the covariant derivative in the equations above is done using the Christoffel connection (22) formed from this metric, and the result (56) would not occur for any other metric. It is also why neutral matter terms formed with $g_{\mu\nu}$ can be appended onto the Lagrangian density (11), because it means that such terms will create divergenceless energy-momentum terms in (52). Now, when charged matter terms containing A_μ are appended onto (11), a few other equations besides (52) acquire additional terms. It is shown in [22] that the divergence of the Einstein equations does not vanish in this case, but instead gives the Lorentz force equation, just as in Einstein-Maxwell theory with sources.

To show that the field equations (48-51) closely approximate electro-vac Einstein-Maxwell theory we will need to make some approximations. The definitions (5,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$ can be inverted to give $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. An expansion in powers of Λ_b^{-1} is derived in Appendix D and confirmed with tetrad methods in [21],

$$N_{(\nu\mu)} = g_{\nu\mu} - 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{2(n-2)}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1} + (f^4)\Lambda_b^{-2}\dots \tag{57}$$

$$N_{[\nu\mu]} = f_{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2} + (f^3)\Lambda_b^{-3/2}\dots. \tag{58}$$

The connection equations (50) can be solved similar to the way that $g_{\mu\nu;\alpha}=0$ is solved to get the Christoffel connection[54]. An expansion in powers of Λ_b^{-1} is derived in [22], confirmed by tetrad methods in [21], and is also stated without derivation in [16],

$$\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \Upsilon^\alpha_{\nu\mu}, \tag{59}$$

$$\Upsilon^\alpha_{(\nu\mu)} = -2\left(f^\tau{}_{(\nu}f_{\mu)}{}^\alpha{}_{;\tau} + f^{\alpha\tau}f_{\tau(\nu;\mu)} + \frac{1}{4(n-2)}((f^{\rho\sigma}f_{\sigma\rho}),{}^\alpha g_{\nu\mu} - 2(f^{\rho\sigma}f_{\sigma\rho}),{}_{(\nu}\delta^\alpha_{\mu)})\right)\Lambda_b^{-1}$$

$$+\,(f^{4\prime})\Lambda_b^{-2}\ldots, \tag{60}$$

$$\Upsilon^\alpha_{[\nu\mu]} = \frac{1}{2}(f_{\nu\mu;}{}^\alpha + f^\alpha{}_{\mu;\nu} - f^\alpha{}_{\nu;\mu})\sqrt{2}\,i\Lambda_b^{-1/2} + (f^{3\prime})\Lambda_b^{-3/2}\ldots. \tag{61}$$

Here $\Gamma^\alpha_{\nu\mu}$ is the Christoffel connection (22). In (57-61) the notation (f^3) and (f^4) refers to terms like $f_{\nu\alpha}f^\alpha{}_\sigma f^\sigma{}_\mu$ and $f_{\nu\alpha}f^\alpha{}_\sigma f^\sigma{}_\rho f^\rho{}_\mu$, and the notation $(f^{3\prime})$ and $(f^{4\prime})$ refers to terms like or $f^\alpha{}_\tau f^\tau{}_\sigma f^\sigma{}_{[\nu;\mu]}$ and $f^\alpha{}_\tau f^\tau{}_\sigma f^\sigma{}_\rho f^\rho{}_{(\nu;\mu)}$. Let us consider worst-case values of these higher order terms relative to the leading order terms. From §7 we know there is an exact electric monopole solution for this theory which approximates a $f^1{}_0 \sim Q/r^2$ field. In geometrized units an elementary charge has

$$Q_e = e\sqrt{\frac{G}{c^4}} = \sqrt{\frac{e^2}{\hbar c}\frac{G\hbar}{c^3}} = \sqrt{\alpha}\,l_P = 1.38\times 10^{-34}cm \tag{62}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant and $l_P = \sqrt{G\hbar/c^3}$ is Planck's constant. If we assume that charged particles retain $f^1{}_0 \sim Q/r^2$ down to the smallest radii probed by high energy particle physics experiments (10^{-17}cm) we have,

$$|f^1{}_0|^2/\Lambda_b \sim (Q_e/(10^{-17})^2)^2/\Lambda_b \sim 10^{-66}. \tag{63}$$

Here $|f^1{}_0|$ is assumed to be in some standard spherical or cartesian coordinate system. If an equation has a tensor term which can be neglected in one coordinate system, it can be neglected in any coordinate system, so it is only necessary to prove it in one coordinate system. The fields at 10^{-17}cm from an elementary charge would be larger than near any macroscopic charged object and would also be larger than the strongest plane-wave fields. Therefore the higher order terms in (57-61) must be $< 10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes. Approximate field equations can be obtained by substituting (57-61) into (46), and we will do this in §4-§5. This gives a set of field equations where $g_{\nu\mu}$ and $f_{\nu\mu}$ are the unknowns instead of $N_{\nu\mu}$. It matters little whether we solve for $g_{\nu\mu}$ and $f_{\nu\mu}$ or for $N_{\nu\mu}$ because they are just related algebraically via (5,24). The advantage of writing equations in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$ is that the close approximation of (52,27) to the ordinary Einstein and Maxwell equations will become apparent.

4. The Symmetric Part of the Field Equations

Here we show that the symmetric part of the field equations contains a close approximation to the ordinary Einstein equations of electro-vac Einstein-Maxwell theory. Subsitituting (57,4) into (52)

$$\begin{aligned}
N_{(\nu\mu)} - \frac{1}{2}\,g_{\nu\mu}N^\rho_\rho &= g_{\nu\mu} - 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{2(n-2)}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1}\\
&\quad - \frac{1}{2}g_{\nu\mu}n + g_{\nu\mu}\left(f^{\rho\sigma}f_{\sigma\rho} - \frac{1}{2(n-2)}nf^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1} + (f^4)\Lambda_b^{-2}\ldots\\
&= -2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{4}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1} - \left(\frac{n}{2}-1\right)g_{\nu\mu} + (f^4)\Lambda_b^{-2}\ldots
\end{aligned}$$

gives approximate Einstein equations,

$$\tilde{G}_{\nu\mu} = 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{4}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right) + \Lambda\left(\frac{n}{2}-1\right)g_{\nu\mu} + (f^4)\Lambda_b^{-1}\ldots. \tag{64}$$

By substituting (59-61,27) into (64,A.4,53) with $\ell = f^{\rho\sigma}f_{\sigma\rho}$,

$$
\begin{aligned}
R_{\nu\mu} &= \tilde{\mathcal{R}}_{(\nu\mu)} - \Upsilon^{\alpha}_{(\nu\mu);\,\alpha} + \Upsilon^{\alpha}_{\alpha(\nu;\mu)} + \Upsilon^{\sigma}_{[\nu\alpha]}\Upsilon^{\alpha}_{[\sigma\mu]} \dots\\
&= \tilde{\mathcal{R}}_{(\nu\mu)} + 2\left(f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\,\tau} + f^{\alpha\tau}f_{\tau(\nu;\,\mu)} + \frac{1}{4(n-2)}(\,\ell,{}^{\alpha}g_{\nu\mu} - 2\,\ell_{,(\nu}\delta^{\alpha}_{\mu)})\right)_{;\,\alpha}\Lambda_b^{-1}\\
&\quad+ \frac{1}{(n-2)}\,\ell_{,(\nu;\,\mu)}\Lambda_b^{-1} - \frac{1}{2}\,(f_{\nu\alpha;}{}^{\sigma} + f^{\sigma}{}_{\alpha;\nu} - f^{\sigma}{}_{\nu;\,\alpha})\,(f_{\sigma\mu;}{}^{\alpha} + f^{\alpha}{}_{\mu;\,\sigma} - f^{\alpha}{}_{\sigma;\,\mu})\,\Lambda_b^{-1} \dots\\
&= \tilde{\mathcal{R}}_{(\nu\mu)} + \left(2f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\tau;\alpha} + 2f^{\alpha\tau}f_{\tau(\nu;\,\mu);\,\alpha} + \frac{1}{2(n-2)}\,\ell,{}^{\alpha}{}_{;\alpha}g_{\nu\mu}\right.\\
&\qquad\qquad \left. - f^{\sigma}{}_{\nu;\alpha}f^{\alpha}{}_{\mu;\sigma} + f^{\sigma}{}_{\nu;\,\alpha}f_{\sigma\mu;}{}^{\alpha} + \frac{1}{2}f^{\sigma}{}_{\alpha;\nu}f^{\alpha}{}_{\sigma;\,\mu}\right)\Lambda_b^{-1}\dots,\\
R &= \tilde{\mathcal{R}}^{\rho}_{\rho} + \left(2f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha} + \frac{n}{2(n-2)}\,\ell,{}^{\alpha}{}_{;\alpha} - f^{\sigma\beta}{}_{;\alpha}f^{\alpha}{}_{\beta;\sigma} + \frac{1}{2}f^{\sigma\beta}{}_{;\,\alpha}f_{\sigma\beta;}{}^{\alpha}\right)\Lambda_b^{-1}\dots\\
&= \tilde{\mathcal{R}}^{\rho}_{\rho} + \left(2f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha} + \frac{n}{2(n-2)}\,\ell,{}^{\alpha}{}_{;\alpha} + \frac{3}{2}f_{[\sigma\beta;\alpha]}f^{[\sigma\beta}{}_{;}{}^{\alpha]}\right)\Lambda_b^{-1}\dots,
\end{aligned}
$$

we see that the Einstein equations (64) can be rewritten in the form

$$
G_{\nu\mu} = 8\pi\tilde{T}_{\nu\mu} + \Lambda\left(\frac{n}{2}-1\right)g_{\nu\mu}, \tag{65}
$$

where

$$
G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2}g_{\nu\mu}R, \tag{66}
$$

$$
\begin{aligned}
8\pi\tilde{T}_{\nu\mu} &= 2\left(f_{\nu}{}^{\sigma}f_{\sigma\mu} - \frac{1}{4}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\\
&\quad+\left(2f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\tau;\alpha} + 2f^{\alpha\tau}f_{\tau(\nu;\,\mu);\alpha} - f^{\sigma}{}_{\nu;\alpha}f^{\alpha}{}_{\mu;\sigma} + f^{\sigma}{}_{\nu;\alpha}f_{\sigma\mu;}{}^{\alpha} + \frac{1}{2}f^{\sigma}{}_{\alpha;\nu}f^{\alpha}{}_{\sigma;\,\mu}\right.\\
&\qquad \left. - g_{\nu\mu}f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha} - \frac{1}{4}g_{\nu\mu}(f^{\rho\sigma}f_{\sigma\rho}),{}^{\alpha}{}_{;\alpha} - \frac{3}{4}g_{\nu\mu}f_{[\sigma\beta;\alpha]}f^{[\sigma\beta}{}_{;}{}^{\alpha]} + (f^4)\right)\Lambda_b^{-1}\dots.
\end{aligned} \tag{67}
$$

In (65-67), $G_{\nu\mu}$, $R_{\nu\mu}$, and R are formed from the Christoffel connection (22), Λ is the small "physical" cosmological constant (4,16), and $\tilde{T}_{\nu\mu}$ is our "effective" energy-momentum tensor. The $(f^4)\Lambda_b^{-1}$ term is $<10^{-66}$ of the ordinary electromagnetic term because of (63). To evaluate the relative contribution of the remaining terms let us consider some worst-case values of $|f^{\mu}{}_{\sigma;\alpha}|$ and $|f^{\mu}{}_{\sigma;\alpha;\beta}|$ accessible to measurement. From §7 we know there is an exact electric monopole solution for this theory which approximates a $f^{1}{}_{0}\sim Q/r^2$ field. If we assume that charged particles retain $f^{1}{}_{0}\sim Q/r^2$ down to very small radii, the values of $|f^{\mu}{}_{\sigma;\alpha}|$ and $|f^{\mu}{}_{\sigma;\alpha;\beta}|$ there would be greater than from any macroscopic monopole field. For the smallest radii probed by high energy particle physics experiments (10^{-17}cm) we have from (14),

$$
|f^{1}{}_{0;1}/f^{1}{}_{0}|^2/\Lambda_b \sim 4/\Lambda_b\,(10^{-17})^2 \sim 10^{-32}, \tag{68}
$$

$$
|f^{1}{}_{0;1;1}/f^{1}{}_{0}|/\Lambda_b \sim 6/\Lambda_b\,(10^{-17})^2 \sim 10^{-32}. \tag{69}
$$

So for electric monopole fields, the extra terms in (67) must be $<10^{-32}$ of the ordinary electromagnetic term. For an electromagnetic plane-wave in a flat background space we have,

$$
A_{\mu} = A\epsilon_{\mu}\sin(k_{\alpha}x^{\alpha}) \quad,\quad \epsilon^{\alpha}\epsilon_{\alpha} = -1 \quad,\quad k^{\alpha}k_{\alpha} = k^{\alpha}\epsilon_{\alpha} = 0, \tag{70}
$$

$$
f_{\nu\mu} = 2A_{[\mu,\nu]} = 2A\epsilon_{[\mu}k_{\nu]}\cos(k_{\alpha}x^{\alpha}). \tag{71}
$$

Here A is the magnitude, k^α is the wavenumber, and ϵ^α is the polarization. Substituting (70,71) into (67), it is easy to see that for flat space all of the extra terms of (67) vanish, and we have as usual,

$$8\pi\,{}^A T_{\nu\mu} \approx\, -A^2\Lambda_b k_\nu k_\mu \cos^2(k_\alpha x^\alpha) = -\frac{A^2\Lambda_b}{2}k_\nu k_\mu(1+\cos(2k_\alpha x^\alpha)). \tag{72}$$

Also, for the highest energy gamma rays known in nature (10^{20}eV) we have from (14),

$$|f^1{}_{0;1}/f^1{}_0|^2/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16}, \tag{73}$$

$$|f^1{}_{0;1;1}/f^1{}_0|/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16}. \tag{74}$$

So for electromagnetic plane-wave fields, even if some of the extra terms in (67) were non-zero because of spatial curvatures, they must still be $< 10^{-16}$ of the ordinary electromagnetic term. Therefore the extra terms in (65-67) must be $< 10^{-16}$ of the ordinary electromagnetic term for even the most extreme worst-case fields accessible to measurement. And we see that (65-67) go to the exact electro-vac Einstein equations in the limit as $\Lambda_b \!\rightarrow\! \infty$.

Now, $G_{\nu\mu}$ in (65) is the ordinary Einstein tensor, so the ordinary contracted Bianchi identity $G^\sigma_{\nu;\,\sigma}\!=\!0$ applies. For the Einstein equations (65) to be compatible, the divergence of these equations should vanish identically, and therefore the divergence of $\tilde T_{\nu\mu}$ from (67) should vanish. This should be expected to occur automatically because we have already shown in (56) that the divergence of the exact Einstein equations (52) vanishes, and because the field equations are derived from a variational principle. Regardless of whether one believes this argument, it can be shown using (27,49,58) that it is indeed true that,

$$\tilde T^\sigma_{\nu;\,\sigma} = \text{at most } \mathcal{O}(\Lambda_b^{-2}). \tag{75}$$

The calculation is rather lengthy so we will omit it. Finally, note that because the divergence of the exact Einstein equations (52) vanishes, our "effective" energy-momentum tensor $\tilde T_{\nu\mu}$ in (65,67) can be augmented by additional energy-momentum tensor contributions caused by non-electromagnetic matter fields. This occurs when additional fields are included in the Lagrangian density as in [22].

5. The Antisymmetric Part of the Field Equations

Here we show that the antisymmetric part of the field equations contain a very close approximation to the ordinary Maxwell equations of electro-vac Einstein-Maxwell theory. Substituting (61,27), into (A.5) gives

$$\begin{aligned}
\tilde{\mathcal{R}}_{[\nu\mu]} &= \Upsilon^\alpha_{[\nu\mu];\alpha} + \mathcal{O}(\Lambda_b^{-3/2})\ldots\\
&= \frac{1}{2}(f_{\nu\mu;}{}^\alpha + f^\alpha{}_{\mu;\nu} - f^\alpha{}_{\nu;\mu})_{;\alpha}\sqrt{2}\,i\Lambda_b^{-1/2}\ldots\\
&= \left(\frac{3}{2}f_{[\nu\mu,\alpha];}{}^\alpha + f^\alpha{}_{\mu;\nu;\alpha} - f^\alpha{}_{\nu;\mu;\alpha}\right)\sqrt{2}\,i\Lambda_b^{-1/2}\ldots\\
&= \left(\frac{3}{2}f_{[\nu\mu,\alpha];}{}^\alpha + 2f^\alpha{}_{\mu;[\nu;\alpha]} - 2f^\alpha{}_{\nu;[\mu;\alpha]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}\ldots.
\end{aligned} \tag{76}$$

From the covariant derivative commutation rule, the definition of the Weyl tensor $C_{\nu\mu\alpha\tau}$, and the Einstein equations $R_{\nu\mu} = -\Lambda g_{\nu\mu} + (f^2)\ldots$ from (65,67) we get

$$2f^\alpha{}_{\nu;[\mu;\alpha]} = R^\tau{}_{\nu\mu\alpha}f^\alpha{}_\tau + R_\tau{}^\alpha{}_{\mu\alpha}f^\tau{}_\nu = \frac{1}{2}R_{\nu\mu\alpha\tau}f^{\alpha\tau} + R^\tau{}_\mu f_{\tau\nu}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(C_{\nu\mu}{}^{\alpha\tau} + \frac{4}{(n-2)}\delta^{[\alpha}_{[\nu}R^{\tau]}_{\mu]} - \frac{2}{(n-1)(n-2)}\delta^{[\alpha}_{[\nu}\delta^{\tau]}_{\mu]}R\right) f_{\alpha\tau} - R^{\tau}{}_{\mu} f_{\nu\tau}\\
&= \frac{1}{2} f^{\alpha\tau} C_{\alpha\tau\nu\mu} + \frac{(n-2)\Lambda}{(n-1)} f_{\nu\mu} + (f^3)\ldots. \qquad (77)
\end{aligned}
$$

Substituting (76,77,58) into the antisymmetric field equations (47) gives

$$
\begin{aligned}
f_{\nu\mu} &= F_{\nu\mu} + \tilde{\mathcal{R}}_{[\nu\mu]}\sqrt{2}\, i\Lambda_b^{-1/2}/2 + (f^3)\Lambda_b^{-1}\ldots \qquad (78)\\
&= F_{\nu\mu} + \left(\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha} + f^{\alpha\tau}C_{\alpha\tau\nu\mu} + \frac{2(n-2)\Lambda}{(n-1)} f_{\nu\mu} + (f^3)\right)\Lambda_b^{-1}\ldots. \qquad (79)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_\tau &= \frac{1}{4} f_{[\nu\mu,\alpha]}\varepsilon_\tau{}^{\nu\mu\alpha}, \quad f_{[\nu\mu,\alpha]} = -\frac{2}{3}\,\theta_\tau\varepsilon^\tau{}_{\nu\mu\alpha}, \qquad (80)\\
\varepsilon_{\tau\nu\mu\alpha} &= (\text{Levi}-\text{Civita tensor}), \qquad (81)\\
C_{\alpha\tau\nu\mu} &= (\text{Weyl tensor}). \qquad (82)
\end{aligned}
$$

In (79) the $F_{\nu\mu}$ term is the ordinary electromagnetic field (23). The $\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha}\Lambda_b^{-1}$ term is divergenceless and appears as $2\theta_{[\mu,\nu]}\Lambda_b^{-1}$ in the dual of $f_{\nu\mu}$. The $(f^3)\Lambda_b^{-1}$ term is $< 10^{-66}$ of $f_{\nu\mu}$ from (63). The $f_{\nu\mu}\Lambda/\Lambda_b$ term is $\sim 10^{-122}$ of $f_{\nu\mu}$ from (16). The Weyl tensor term might be expected to have the largest observable values near the Schwarzschild radius, $r_s = 2Gm/c^2$, of black holes, where $C_{\nu\mu\alpha\tau}$ takes on values around r_s/r^3. However, since the lightest black holes have the smallest Schwarzschild radius, they will create the largest value of $r_s/r_s^3 = 1/r_s^2$. The lightest black hole that we can expect to observe would be of about one solar mass, where from (14),

$$
\frac{C_{trtr}}{\Lambda_b} \sim \frac{1}{\Lambda_b r_s^2} = \frac{1}{\Lambda_b}\left(\frac{c^2}{2Gm_\odot}\right)^2 \sim 10^{-77}. \qquad (83)
$$

So even in the most extreme worst-cases accessible to measurement, the last three terms in (79) are all $< 10^{-66}$ of $f_{\nu\mu}$. And we set that (79) gives exactly $f_{\nu\mu} = F_{\nu\mu}$ in the limit as $\Lambda_b \to \infty$.

Taking the divergence of (79) using (27), the divergenceless term $\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha}\Lambda_b^{-1}$ falls out and we get an extremely close approximation to Maxwell's equations,

$$
\begin{aligned}
F_{\nu\mu;}{}^{\nu} &= \left[(f^{\tau\alpha}C_{\alpha\tau\nu\mu});^{\nu} + (f^{3\prime})\right]\Lambda_b^{-1}\ldots, \qquad (84)\\
F_{[\nu\mu,\sigma]} &= 0. \qquad (85)
\end{aligned}
$$

As usual, Faraday's law (85) is just an identity which follows from the definition (23). The extra terms in Ampere's law (84) are $< 10^{-66}$ of the primary terms because this is true for (79). In most so-called "exact" equations in physics, there are really many known corrections due to QED and other effects which are ignored because they are too small to measure. We should emphasize that the extra terms in (84) are at least 50 orders of magnitude smaller than known corrections to Maxwell's equations which are routinely ignored[55]. And we see that (84,85) go to the exact electro-vac Maxwell equations in the limit as $\Lambda_b \to \infty$. Of course we are just considering the electro-vac case, so (84) has no source term. As in electro-vac Einstein-Maxwell theory, the lack of a source term does not preclude the existence of charges, as evidenced by the exact electric monopole solution derived in §7. To get a source term in Ampere's law (84), charged matter terms must be included in the Lagrangian density as in [22].

The divergencless term $\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha}$ of (79) should also be expected to be $< 10^{-32}$ of $f_{\nu\mu}$ from (68,69,80). However, we need to consider the possibility where θ_τ changes

extremely rapidly, so let us consider the "dual" part of (79). Taking the curl of (79), the $F_{\nu\mu}$ term falls out from (23) and we have

$$f_{[\nu\mu,\sigma]} = \left(\theta_{\tau;\alpha;[\sigma}\,\varepsilon_{\nu\mu]}{}^{\tau\alpha} + (f^{\alpha\tau}C_{\alpha\tau[\nu\mu})_{,\sigma]} + \frac{2(n-2)\Lambda}{(n-1)}f_{[\nu\mu,\sigma]} + (f^{3\prime})\right)\Lambda_b^{-1}\dots. \tag{86}$$

Contracting this with $\Lambda_b\varepsilon^{\rho\sigma\nu\mu}/2$ and using (80) gives,

$$2\Lambda_b\theta^\rho = -\,2\,\theta^{[\rho}{}_;{}^{\sigma]}{}_{;\sigma} + \frac{1}{2}\varepsilon^{\rho\sigma\nu\mu}(f^{\alpha\tau}C_{\alpha\tau[\nu\mu})_{,\sigma]} + \frac{4(n-2)\Lambda}{(n-1)}\,\theta^\rho + (f^{3\prime})\dots \tag{87}$$

Using $\theta^\sigma{}_{;\sigma} = 0$ from (80) and the covariant derivative commutation rule, the Einstein equations $R_{\nu\mu} = -\Lambda g_{\nu\mu} + (f^2)\dots$ from (65,67) give $\theta^\sigma{}_{;\rho;\sigma} = R_{\sigma\rho}\theta^\sigma = -\theta_\rho\Lambda + (f^{3\prime})\dots$, and we get something similar to the Proca equation[56, 57],

$$2\Lambda_b\theta_\rho = -\theta_{\rho;\,\sigma;}{}^\sigma + \frac{1}{2}\varepsilon_\rho{}^{\sigma\nu\mu}(f^{\alpha\tau}C_{\alpha\tau[\nu\mu})_{,\sigma]} + \frac{(3n-7)\Lambda}{(n-1)}\,\theta_\rho + (f^{3\prime})\dots. \tag{88}$$

Here the $\Lambda\theta_\rho$ term can certainly be ignored from (16), and the $(f^{3\prime})$ term can probably be ignored in the weak field limit. The Weyl tensor term can be ignored if one assumes a flat background, although Proca waves might significantly perturb the background if they exist, so this is a rather big assumption. If we do ignore the last three terms, this equation has the trivial solution $\theta_\rho \approx 0$. If $\Lambda_z > 0$, $\Lambda_b < 0$ as with supersymmetry, wavelike solutions to (88) cannot exist. If $\Lambda_z < 0$, $\Lambda_b > 0$ as in (14,15), wavelike solutions could possibly exist, and in a flat background space they would be of the form[56]

$$\theta_\rho = \theta\epsilon_\rho\sin(k_\alpha x^\alpha) \quad,\quad k_\alpha k^\alpha = 2\Lambda_b \quad,\quad \epsilon_\alpha\epsilon^\alpha = -1 \quad,\quad k_\alpha\epsilon^\alpha = 0, \tag{89}$$

$$\omega = \sqrt{2\Lambda_b + \mathbf{k}^2} \quad,\quad k_\alpha = (\omega, \mathbf{k})\,. \tag{90}$$

Here θ is the magnitude, k^α is the wavenumber, ϵ^α is the polarization, and ω is the frequency. Substituting (89,90) into (79,67)

$$\begin{aligned}
f^{\nu\sigma} &= \varepsilon^{\nu\sigma\rho\tau}\theta_{[\rho,\tau]}\Lambda_b^{-1} = \theta\varepsilon^{\nu\sigma\rho\tau}\epsilon_{[\rho}k_{\tau]}\cos(k_\alpha x^\alpha)\Lambda_b^{-1},\\
f^{\nu\sigma}f_{\sigma\mu} &= \varepsilon^{\nu\sigma\rho\tau}\theta_{[\rho,\tau]}\varepsilon_{\sigma\mu\lambda\beta}\theta^{[\lambda}{}_;{}^{\beta]}\Lambda_b^{-2} = 6\delta^{[\nu}_\mu\delta^\rho_\lambda\delta^{\tau]}_\beta\theta_{\rho,\tau}\theta^{[\lambda}{}_;{}^{\beta]}\Lambda_b^{-2}\\
&= 2(\delta^\nu_\mu\theta_{\lambda,\beta}\theta^{[\lambda}{}_;{}^{\beta]} + \theta_{\mu,\lambda}\theta^{[\lambda}{}_;{}^{\nu]} + \theta_{\beta,\mu}\theta^{[\nu}{}_;{}^{\beta]})\Lambda_b^{-2}\\
&= \theta^2(\delta^\nu_\mu\epsilon_\lambda\epsilon^\lambda k_\beta k^\beta - \epsilon_\mu\epsilon^\nu k_\lambda k^\lambda - \epsilon_\beta\epsilon^\beta k_\mu k^\nu)\cos^2(k_\alpha x^\alpha)\Lambda_b^{-2}\\
&= \theta^2(-2\Lambda_b\delta^\nu_\mu - 2\Lambda_b\epsilon_\mu\epsilon^\nu + k_\mu k^\nu)\cos^2(k_\alpha x^\alpha)\Lambda_b^{-2},\\
f^{\nu\sigma}f_{\sigma\nu} &= -\,4\theta^2\cos^2(k_\alpha x^\alpha)\Lambda_b^{-1},\\
8\pi\,{}^\theta T_{\nu\mu} &\approx 2\theta^2(-2\Lambda_b g_{\nu\mu} - 2\Lambda_b\epsilon_\nu\epsilon_\mu + k_\nu k_\mu + \Lambda_b g_{\nu\mu})\cos^2(k_\alpha x^\alpha)\Lambda_b^{-2} + 0 + 0 + 0\\
&\quad + 2\theta^2(2\Lambda_b g_{\nu\mu} + 2\Lambda_b\epsilon_\nu\epsilon_\mu - k_\nu k_\mu)\sin^2(k_\alpha x^\alpha)\Lambda_b^{-2} - 2\theta^2 k_\nu k_\mu\sin^2(k_\alpha x^\alpha)\Lambda_b^{-2}\\
&\quad + 0 + 4\theta^2 g_{\nu\mu}(\sin^2(k_\alpha x^\alpha) - \cos^2(k_\alpha x^\alpha))\Lambda_b^{-1} - 2\theta^2 g_{\nu\mu}\sin^2(k_\alpha x^\alpha)\Lambda_b^{-1}
\end{aligned}$$

gives the energy-momentum tensor

$$8\pi\,{}^\theta T_{\nu\mu} \approx -\frac{\theta^2}{\Lambda_b}\,[k_\nu k_\mu\Lambda_b^{-1} - (3k_\nu k_\mu\Lambda_b^{-1} - 6g_{\nu\mu} - 4\,\epsilon_\nu\epsilon_\mu)\cos(2k_\alpha x^\alpha)]. \tag{91}$$

Here we find that $\left\langle{}^\theta T_{00}\right\rangle < 0$, an indication that the theory might allow negative energy waves, often called "ghosts". However, unlike similar theories[31, 32, 33], this theory avoids ghosts in an unusual way. Recall that this theory is the original Einstein-Schrödinger theory, but with a $\Lambda_z g_{\mu\nu}$ in the field equations to account for zero-point fluctuations, and $\Lambda_z = -C_z\omega_c^4 l_P^2$ from (14,15) is finite only because of a cutoff frequency

$\omega_c \sim 1/l_P$ from (13). From these equations and (88), Proca waves would be cut off because they would have a minimum frequency

$$\omega_{Proca} = \sqrt{2\Lambda_b} = \sqrt{-2\Lambda_z} = \sqrt{2C_z}\,\omega_c^2 l_P > \omega_c. \tag{92}$$

Whether the cutoff of zero-point fluctuations is caused by a discreteness, uncertainty or foaminess of spacetime near the Planck length[51, 47, 48, 50, 52] or by some other effect, the same ω_c which cuts off Λ_z should also cut off Proca waves in this theory. So we should expect to observe only the trivial solution $\vartheta_\rho \approx 0$ to (88) and no ghosts. Comparing ω_{Proca} and ω_c from above, we see that this argument only applies if

$$\omega_c > \frac{1}{l_P\sqrt{2C_z}}. \tag{93}$$

Here C_z is defined by (15), and the inequality is satisfied for this theory when ω_c and C_z are chosen as in (13,14) to be consistent with a cosmological constant caused by zero-point fluctuations. Since the prediction of negative energy waves would probably be inconsistent with reality, this theory should be approached cautiously when considering it with values of ω_c and C_z which do not satisfy (93).

Finally, if we fully renormalize with $\omega_c \to \infty$ as in quantum electrodynamics, then $\Lambda_b \to \infty$ and $\omega_{Proca} \to \infty$, so the potential ghost goes away completely. In the limit $\omega_c \to \infty$ our theory becomes exactly Einstein-Maxwell theory. However, the theory would still be much different than Einstein-Maxwell theory from the standpoint of quantization. In any attempt to quantize this theory, the cutoff frequency ω_c would need to be the same cutoff which is taken to infinity during renormalization. For example, Pauli-Villars masses would probably go as $M = \hbar\omega_c$ if Pauli-Villars renormalization was used. Since Λ_b and Λ_z in the Lagrangian density go as ω_c^4, quantization and renormalization would certainly need to be done a bit different than usual. Also, because ω_{Proca} goes as ω_c^2, Proca waves would not represent a ghost from the standpoint of quantization.

6. The Einstein-Infeld-Hoffmann equations of motion

Here we derive the Lorentz force from the theory using the Einstein-Infeld-Hoffmann (EIH) method[23]. For Einstein-Maxwell theory, the EIH method allows the equations of motion to be derived directly from the electro-vac field equations. For neutral particles the method has been verified to Post-Newtonian order[23], and in fact it was the method first used to derive the Post-Newtonian equations of motion[59]. For charged particles the method has been verified to Post-Coulombian order[24, 60, 61], (see also Appendix F) meaning that it gives the same result as the Darwin Lagrangian[55]. The EIH method is valuable because it does not require any additional assumptions, such as the postulate that neutral particles follow geodesics, or the *ad hoc* inclusion of matter terms in the Lagrangian density. When the EIH method was applied to the original Einstein-Schrödinger theory, no Lorentz force was found between charged particles[25, 26]. The basic difference between our case and [25, 26] is that our effective energy-momentum tensor (67) contains the familiar term $f_\nu{}^\sigma f_{\sigma\mu} - (1/4)g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}$. This term appears because we assumed $\Lambda_b \neq 0$, and because of our metric definition (5) and (57). With this term, the EIH method predicts the same Lorentz force as it does for electro-vac Einstein-Maxwell theory. Also, it happens that the extra terms in our approximate Einstein and Maxwell equations (65-67,84,85) cause no contribution beyond the Lorentz force, to Newtonian/Coulombian order. The basic reason for the null result of [25, 26] is that they assumed $\Lambda_b = 0$ and

$g_{\mu\nu}=N_{(\mu\nu)}$, so that every term in their effective energy-momentum tensor has "extra derivatives"[62]. For the same reason that [25, 26] found no Lorentz force, the extra derivative terms in our effective energy-momentum tensor (67) cause no contribution to the equations of motion.

The exact Lorentz force equation can be derived for this theory by including charged matter terms in the Lagrangian density[22]. Here we derive the Lorentz force using the EIH method because it requires no assumptions about matter terms, and also to show definitely that the well known negative result of [25, 26] for the unmodified Einstein-Schrödinger theory does not apply to the present theory. We will only cover the bare essentials of the EIH method which are necessary to derive the Lorentz force, and the references above should be consulted for a more complete explanation. We will also only calculate the equations of motion to Newtonian/Coulombian order, because this is the order where the Lorentz force first appears.

With the EIH method, one does not just find equations of motion, but rather one finds approximate solutions $g_{\mu\nu}$ and $f_{\mu\nu}$ of the field equations which correspond to a system of two or more particles. These approximate solutions will in general contain $1/r^p$ singularities, and these are considered to represent particles. It happens that acceptable solutions to the field equations can only be found if the motions of these singularities are constrained to obey certain equations of motion. The assumption is that these approximate solutions for $g_{\mu\nu}$ and $f_{\mu\nu}$ should approach exact solutions asymptotically, and therefore the motions of the singularities should approximate the motions of exact solutions. Any event horizon or other unusual feature of exact solutions at small radii is irrelevant because the singularities are assumed to be separated by much larger distances, and because the method relies greatly on surface integrals done at large distances from the singularities. Some kind of exact Reissner-Nordström-like solution should probably exist in order for the EIH method to make sense, and the electric monopole solution in §7 fills this role in our case. However, exact solutions are really only used indirectly to identify constants of integration.

The EIH method assumes the "slow motion approximation", meaning that $v/c\ll 1$. The fields are expanded in the form[23, 24, 60, 61],

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} - \eta_{\mu\nu}\eta^{\sigma\rho}\gamma_{\sigma\rho}/2, \tag{94}$$
$$\gamma_{00} = {}_2\gamma_{00}\lambda^2 + {}_4\gamma_{00}\lambda^4 \ldots \tag{95}$$
$$\gamma_{0k} = {}_3\gamma_{0k}\lambda^3 + {}_5\gamma_{0k}\lambda^5 \ldots \tag{96}$$
$$\gamma_{ik} = {}_4\gamma_{ik}\lambda^4 \ldots \tag{97}$$
$$A_0 = {}_2A_0\lambda^2 + {}_4A_0\lambda^4 \ldots \tag{98}$$
$$A_k = {}_3A_k\lambda^3 + {}_5A_k\lambda^5 \ldots \tag{99}$$
$$f_{0k} = {}_2f_{0k}\lambda^2 + {}_4f_{0k}\lambda^4 \ldots \tag{100}$$
$$f_{ik} = {}_3f_{ik}\lambda^3 + {}_5f_{ik}\lambda^5 \ldots \tag{101}$$

where $\lambda \sim v/c$ is the expansion parameter, the order of each term is indicated with a left subscript[25], $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and Latin indices run from 1-3. The field $\gamma_{\mu\nu}$ (often called $\bar{h}_{\mu\nu}$ in other contexts) is used instead of $g_{\mu\nu}$ only because it simplifies the calculations. Because $\lambda \sim v/c$, when the expansions are substituted into the Einstein and Maxwell equations, a time derivative counts the same as one higher order in λ. The general procedure is to substitute the expansions, and solve the resulting field equations order by order in λ, continuing to higher orders until a desired level of accuracy is achieved. At each order in λ, one of the ${}_l\gamma_{\mu\nu}$ terms and

one of the ${}_l f_{\mu\nu}$ terms will be unknowns, and the equations will involve known results from previous orders because of the nonlinearity of the Einstein equations.

The expansions (95-101) use only alternate powers of λ essentially because the Einstein and Maxwell equations are second order differential equations[59], although for higher powers of λ, all terms must be included to predict radiation[24, 60, 61]. Because $\lambda \sim v/c$, the expansions have the magnetic components A_k and f_{ik} due to motion at one order higher in λ than the electric components A_0 and f_{0i}. As in [24, 60, 61], f_{0k} and f_{ik} have even and odd powers of λ respectively. This is the opposite of [25, 26] because we are assuming a direct definition of the electromagnetic field (24,58,79,23) instead of the dual definition $f^{\alpha\rho} = \varepsilon^{\alpha\rho\sigma\mu} N_{[\sigma\mu]}/2$ assumed in [25, 26].

The field equations are assumed to be of the standard form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad \text{where} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}, \tag{102}$$

or equivalently

$$R_{\mu\nu} = 8\pi S_{\mu\nu} \quad \text{where} \quad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}. \tag{103}$$

However, with the EIH method we must solve a sort of quasi-Einstein equations,

$$0 = \breve{G}_{\mu\nu} - 8\pi \breve{T}_{\mu\nu}, \tag{104}$$

where

$$\breve{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} R_{\alpha\beta}, \qquad\qquad \breve{T}_{\mu\nu} = S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} S_{\alpha\beta}. \tag{105}$$

Here the use of $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ is not an approximation because (103) implies (104) whether $\breve{G}_{\mu\nu}$ and $\breve{T}_{\mu\nu}$ are defined with $\eta_{\mu\nu}$ or $g_{\mu\nu}$. Note that the references use many different notations in (104): instead of $\breve{G}_{\mu\nu}$ others use $\Pi_{\mu\nu}/2 + \Lambda_{\mu\nu}$, $\Phi_{\mu\nu}/2 + \Lambda_{\mu\nu}$ or [LS$:\mu\nu$] and instead of $8\pi \breve{T}_{\mu\nu}$ others use $-2S_{\mu\nu}$, $-\Lambda'_{\mu\nu}$, $-\Lambda_{\mu\nu}$ or [RS$:\mu\nu$].

The equations of motion result as a condition that the field equations (104) have acceptable solutions. In the language of the EIH method, acceptable solutions are those that contain only "pole" terms and no "dipole" terms, and this can be viewed as a requirement that the solutions should resemble Reissner-Nordström solutions asymptotically. To express the condition of solvability we must consider the integral of the field equations (104) over 2D surfaces S surrounding each singularity,

$${}_l C_\mu = \frac{1}{2\pi} \int^S ({}_l \breve{G}_{\mu k} - 8\pi\, {}_l \breve{T}_{\mu k}) n_k dS. \tag{106}$$

Here n_k is the surface normal and l is the order in λ. Assuming that the divergence of the Einstein equations (102) vanishes, and that (104) has been solved to all previous orders, it can be shown[23] that in the current order

$$({}_l \breve{G}_{\mu k} - 8\pi\, {}_l \breve{T}_{\mu k})_{|k} = 0. \tag{107}$$

Here and throughout this section "|" represents ordinary derivative[23]. From Green's theorem, (107) implies that ${}_l C_\mu$ in (106) will be independent of surface size and shape[23]. The condition for the existence of an acceptable solution for ${}_4\gamma_{ik}$ is simply

$${}_4 C_i = 0, \tag{108}$$

and these are also our three $\mathcal{O}(\lambda^4)$ equations of motion[23]. The C_0 component of (106) causes no constraint on the motion[23] so we only need to calculate $\breve{G}_{ik}$ and $\breve{T}_{ik}$.

At this point let us introduce a Lemma from [23] which is derived from Stokes's theorem. This Lemma states that

$$\int^{S}\mathcal{F}_{(\dots)kl|l}n_k dS=0 \quad \text{if} \quad \mathcal{F}_{(\dots)kl}=-\mathcal{F}_{(\dots)lk}, \tag{109}$$

where $\mathcal{F}_{(\dots)kl}$ is any antisymmetric function of the coordinates, n_k is the surface normal, and S is any closed 2D surface which may surround a singularity. The equation ${}_4C_i=0$ is a condition for the existence of a solution for ${}_4\gamma_{ik}$ because ${}_4\gamma_{ik}$ is found by solving the $\mathcal{O}(\lambda^4)$ field equations (104), and ${}_4C_i$ is the integral (106) of these equations. However, because of the Lemma (109), it happens that the ${}_4\gamma_{ik}$ terms in ${}_4\breve{G}_{ik}$ integrate to zero in (106), so that ${}_4C_i$ is actually independent of ${}_4\gamma_{ik}$. In fact it is a general rule that C_i for one order can be calculated using only results from previous orders[23], and this is a crucial aspect of the EIH method. Therefore, the calculation of the $\mathcal{O}(\lambda^4)$ equations of motion (108) does not involve the calculation of ${}_4\gamma_{ik}$, and we will see below that it also does not involve the calculation of ${}_3f_{ik}$ or ${}_4f_{0k}$.

The ${}_4\breve{G}_{ik}$ contribution to (106) is derived in [23]. For two particles with masses m_1, m_2 and positions ξ^i_1, ξ^i_2, the $\mathcal{O}(\lambda^4)$ term from the integral over the first particle is

$$\overset{\breve{G}}{{}_4}C_i=\frac{1}{2\pi}\int^{1}{}_4\breve{G}_{ik}n_k dS=-4\left\{m_1\ddot{\xi}^i_1-m_1m_2\frac{\partial}{\partial\xi^i_1}\left(\frac{1}{r}\right)\right\}, \tag{110}$$

where

$$r=\sqrt{(\xi^s_1-\xi^s_2)(\xi^s_1-\xi^s_2)}\,. \tag{111}$$

If there is no other contribution to (106), then (108) requires that $\overset{\breve{G}}{{}_4}C_i\!=\!0$ in (110), and the particle acceleration will be proportional to a $\nabla(m_1m_2/r)$ Newtonian gravitational force. These are the EIH equations of motion for vacuum general relativity to $\mathcal{O}(\lambda^4)$, or Newtonian order.

Because our effective energy-momentum tensor (67) is quadratic in $f_{\mu\nu}$, and the expansions (95-101) begin with λ^2 terms, the $\mathcal{O}(\lambda^2)-\mathcal{O}(\lambda^3)$ calculations leading to (110) are unaffected by the addition of (67) to the vacuum field equations. However, the $8\pi\,{}_4\breve{T}_{ik}$ contribution to (106) will add to the ${}_4\breve{G}_{ik}$ contribution. To calculate this contribution, we will assume that our singularities in $f_{\nu\mu}$ are simple moving Coulomb potentials, and that $\theta^\rho=0$, $\Lambda=0$. Then from (79,100-101) we see that ${}_2F_{0k}={}_2f_{0k}$, and from inspection of the extra terms in our Maxwell equations (84,85) and Proca equation (88), we see that these equations are both solved to $\mathcal{O}(\lambda^3)$. Because (67) is quadratic in $f_{\mu\nu}$, we see from (100-101) that only ${}_2f_{0k}$ can affect the $\mathcal{O}(\lambda^4)$ equations of motion. Including only ${}_2f_{0k}$, our $f_{\mu\nu}$ is then a sum of two Coulomb potentials with charges Q_1, Q_2 and positions ξ^i_1, ξ^i_2 of the form

$$ {}_2A_\mu=({}_2\varphi,0,0,0) \quad,\quad {}_2f_{0k}=2\,{}_2A_{[k|0]}=-\,{}_2\varphi_{|k}, \tag{112}$$

$$ {}_2\varphi=\psi^1+\psi^2 \quad,\quad \psi^1=Q_1/r_1 \quad,\quad \psi^2=Q_2/r_2, \tag{113}$$

$$ r_p=\sqrt{(x^s-\xi^s_p)(x^s-\xi^s_p)} \quad,\quad p=1...2\,. \tag{114}$$

Because our effective energy momentum tensor (67) is quadratic in both $f_{\mu\nu}$ and $g_{\mu\nu}$, and the expansions (95-101) start at λ^2 in both of these quantitites, no gravitational-electromagnetic interactions will occur at $\mathcal{O}(\lambda^4)$. This allows us to replace covariant derivatives with ordinary derivatives, and $g_{\nu\mu}$ with $\eta_{\nu\mu}$ in (67). This also allows us to replace $\breve{T}_{\mu\nu}$ from (104,105) with (67),

$$\breve{T}_{\mu\nu}=S_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}S_{\alpha\beta}\approx\tilde{S}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\tilde{S}_{\alpha\beta}=\tilde{T}_{\mu\nu}. \tag{115}$$

Therefore, from (67) we have

$$8\pi\breve{T}_{\nu\mu} = 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{4}\eta_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right) \tag{116}$$

$$+\left(2f^\tau{}_{(\nu}f_{\mu)}{}^\alpha{}_{|\tau|\alpha} + 2f^{\alpha\tau}f_{\tau(\nu|\mu)|\alpha} - f^\sigma{}_{\nu|\alpha}f^\alpha{}_{\mu|\sigma} + f^\sigma{}_{\nu|\alpha}f_{\sigma\mu,}{}^\alpha + \frac{1}{2}f^\sigma{}_{\alpha|\nu}f^\alpha{}_{\sigma|\mu}\right.$$
$$\left. -\eta_{\nu\mu}f^{\tau\beta}f_\beta{}^\alpha{}_{|\tau|\alpha} - \frac{1}{4}\eta_{\nu\mu}(f^{\rho\sigma}f_{\sigma\rho})_|{}^\alpha{}_{|\alpha} - \frac{3}{4}\eta_{\nu\mu}f_{[\sigma\beta|\alpha]}f^{[\sigma\beta}{}_|{}^{\alpha]} + (f^4)\right)\Lambda_b^{-1}. \tag{117}$$

This can be simplified by keeping only $\mathcal{O}(\lambda^4)$ terms. The terms $2f^\tau{}_{(\nu}f_{\mu)}{}^\alpha{}_{|\tau|\alpha}$ and $-\eta_{\nu\mu}f^{\tau\beta}f_\beta{}^\alpha{}_{|\tau|\alpha}$ vanish because (112) satisfies Ampere's law to $\mathcal{O}(\lambda^2)$. The term $-(3/4)\eta_{\nu\mu}f_{[\sigma\beta|\alpha]}f^{[\sigma\beta}{}_|{}^{\alpha]}$ vanishes because (112) satisfies $f_{[\sigma\beta|\alpha]} = 2A_{[\beta|\sigma|\alpha]} = 0$. Also, since time derivatives count the same as a higher order in λ, we can remove the term $-f^\sigma{}_{s|\alpha}f^\alpha{}_{m|\sigma} = -f^0{}_{s|0}f^0{}_{m|0}$, and we can change some of the summations over Greek indices to summations over Latin indices. The (f^4) term will be $\mathcal{O}(\lambda^8)$ so it can obviously be eliminated. And as mentioned above, only ${}_2f_{0k}$ contributes at $\mathcal{O}(\lambda^4)$. Applying these results, and dropping the order subscripts to reduce the clutter, the spatial part of (116) becomes,

$$8\pi\,{}_4\breve{T}_{sm} = 2\left(f_s{}^0 f_{0m} - \frac{1}{2}\eta_{sm}f^{r0}f_{0r}\right)$$
$$+\left(2f^{a0}f_{0(s|m)|a} + f^0{}_{s|a}f_{0m|}{}^a + f^0{}_{a|s}f^a{}_{0|m} - \frac{1}{2}\eta_{sm}(f^{r0}f_{0r})_|{}^a{}_{|a}\right)\Lambda_b^{-1} \tag{118}$$
$$= -2\left(f_{0s}f_{0m} + \frac{1}{2}\eta_{sm}f_{0r}f_{0r}\right)$$
$$+\left(2f_{0a}f_{0(s|m)|a} - f_{0s|a}f_{0m|a} + f_{0a|s}f_{0a|m} + \frac{1}{2}\eta_{sm}(f_{0r}f_{0r})_{|a|a}\right)\Lambda_b^{-1}. \tag{119}$$

Note that ${}_2\varphi$ from (113) obeys Gauss's law,

$$\varphi_{|a|a} = 0. \tag{120}$$

Substituting (112) into (119) and using (120) gives

$$8\pi\,{}_4\breve{T}_{sm} = -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right)$$
$$+\left(2\varphi_{|a}\varphi_{|s|m|a} - \varphi_{|s|a}\varphi_{|m|a} + \varphi_{|a|s}\varphi_{|a|m} + \frac{1}{2}\eta_{sm}(\varphi_{|r}\varphi_{|r})_{|a|a}\right)\Lambda_b^{-1} \tag{121}$$
$$= -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right)$$
$$-(\varphi_{|s}\varphi_{|a|m} + \varphi_{|r}\varphi_{|r|s}\eta_{am})_{|a}\Lambda_b^{-1} + (\varphi_{|a}\varphi_{|s|m} + \varphi_{|r}\varphi_{|r|a}\eta_{sm})_{|a}\Lambda_b^{-1}$$
$$= -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right) - 2(\varphi_{|[s}\varphi_{|a]|m} + \varphi_{|r}\varphi_{|r|[s}\eta_{a]m})_{|a}\Lambda_b^{-1}. \tag{122}$$

From (109) we see that the second group of terms in (122) integrates to zero in (106), and therefore it can have no effect on the equations of motion. The first group of terms in (122) is what one gets with ordinary electro-vac Einstein-Maxwell theory[24, 60, 61], so at this stage we have effectively proven that the theory predicts a Lorentz force.

For completeness we will finish the derivation. First, we see from (122,120) that ${}_4\breve{T}_{sm|s} = 0$. This is to be expected because of (75,107), and it means that the

$8\pi\,{}_4\breve{T}_{sm}$ contribution to the surface integral (106) will be independent of surface size and shape. This also means that only contributions from 1/distance^2 terms such as η_{sm}/r^2 or $x_s x_m/r^4$ can contribute to (106). The integral over a term with any other distance-dependence would necessarily depend on the surface radius, and therefore we know beforehand that it must vanish or cancel with other similar terms[23]. Now, $\varphi_{|i} = \psi^1_{|i} + \psi^2_{|i}$ from (113). Because $\psi^1_{|i}$ and $\psi^2_{|i}$ both go as 1/distance^2, but are in different locations, it is clear from (122) that contributions can only come from cross terms between the two. Including only these terms gives,

$$8\pi\,{}_4\breve{T}^c_{sm} = -2\left(\psi^1_{|s}\psi^2_{|m} + \psi^2_{|s}\psi^1_{|m} + \eta_{sm}\psi^1_{|r}\psi^2_{|r}\right). \tag{123}$$

Some integrals we will need can be found in [23]. With $\psi = 1/\sqrt{x^s x^s}$ we have,

$$\frac{1}{4\pi}\int^0 \psi_{|m} n_m dS = -1 \quad , \quad \frac{1}{4\pi}\int^0 \psi_{|a} n_m dS = -\frac{1}{3}\delta_{am}. \tag{124}$$

Using (123,124,113) and integrating over the first particle we get,

$$\frac{1}{2\pi}\int^1 \left[-8\pi\breve{T}_{sm}\right] n_m dS = \frac{1}{2\pi}\int^1 2\left(\psi^1_{|s}\psi^2_{|m} + \psi^2_{|s}\psi^1_{|m} + \eta_{sm}\psi^1_{|r}\psi^2_{|r}\right) n_m dS \tag{125}$$

$$= 4Q_1\psi^2_{|s}(\xi_1)\left(-\frac{1}{3} - 1 + \frac{1}{3}\right) = -4Q_1\psi^2_{|s}(\xi_1). \tag{126}$$

Using (108,106,126,110,113) we get

$$0 = {}_4C_i = -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2\frac{\partial}{\partial\xi^i_1}\left(\frac{1}{r}\right)\right\} - 4Q_1\psi^2_{|i}(\xi_1) \tag{127}$$

$$= -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2\frac{\partial}{\partial\xi^i_1}\left(\frac{1}{r}\right)\right\} - 4Q_1\frac{\partial}{\partial\xi^i_1}\left(\frac{Q_2}{r}\right)$$

$$= -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2\frac{\partial}{\partial\xi^i_1}\left(\frac{1}{r}\right) + Q_1 Q_2\frac{\partial}{\partial\xi^i_1}\left(\frac{1}{r}\right)\right\}, \tag{128}$$

where

$$r = \sqrt{(\xi^s_1 - \xi^s_2)(\xi^s_1 - \xi^s_2)}. \tag{129}$$

These are the EIH equations of motion for this theory to $\mathcal{O}(\lambda^4)$, or Newtonian/Coulombian order. These equations of motion clearly exhibit the Lorentz force, and in fact they match the $\mathcal{O}(\lambda^4)$ equations of motion of Einstein-Maxwell theory.

7. An exact electric monopole solution

Here we derived an exact charged solution for this theory which closely approximates the Reissner-Nordström solution of Einstein-Maxwell theory. A MAPLE program[9] which checks the solution is also available. It can be shown[63] that the assumption of spherical symmetry allows the fundamental tensor to be written in the following form

$$N_{\nu\mu} = \begin{pmatrix} \gamma & -w & 0 & 0 \\ w & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & r^2 v\sin\theta \\ 0 & 0 & -r^2 v\sin\theta & -\beta\sin^2\theta \end{pmatrix}. \tag{130}$$

Both [63] and [64] assume this form with $\beta = r^2, v = 0$ to derive a solution to the original Einstein-Schrödinger field equations which looks similar to a charged mass, but with some problems. Here we will derive a solution to the modified field equations

(48-51) which is much closer to the Reissner-Nordström solution[65, 66] of ordinary electro-vac Einstein-Maxwell theory. We will follow a similar procedure to [63, 64] but will use coordinates $x_0, x_1, x_2, x_3 = ct, r, \theta, \phi$ instead of $x_1, x_2, x_3, x_4 = r, \theta, \phi, ct$. We also use the variables $a = 1/\alpha,\ b = \gamma\alpha,\ \check{s} = -w$, which allow a simpler solution than the variables α, γ, w. This gives

$$N_{\nu\mu} = \begin{pmatrix} ab & \check{s} & 0 & 0 \\ -\check{s} & -1/a & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{131}$$

$$N^{\dashv\mu\nu} = \begin{pmatrix} 1/ad & \check{s}/d & 0 & 0 \\ -\check{s}/d & -ab/d & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, \tag{132}$$

$$\sqrt{-N} = \sqrt{d}\, r^2 \sin\theta, \tag{133}$$

where

$$d = b - \check{s}^2. \tag{134}$$

From (132,133) and the definitions (5,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$ we get

$$g^{\nu\mu} = \frac{1}{\check{c}}\begin{pmatrix} 1/ad & 0 & 0 & 0 \\ 0 & -ab/d & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, \quad f^{\nu\mu} = \frac{\Lambda_b^{1/2}}{\sqrt{2}\, i\, \check{c}}\begin{pmatrix} 0 & -\check{s}/d & 0 & 0 \\ \check{s}/d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{135}$$

$$g_{\nu\mu} = \check{c}\begin{pmatrix} ad & 0 & 0 & 0 \\ 0 & -d/ab & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \quad f_{\nu\mu} = \frac{\Lambda_b^{1/2}}{\sqrt{2}\, i\, \check{c}}\begin{pmatrix} 0 & \check{s} & 0 & 0 \\ -\check{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{136}$$

$$\sqrt{-g} = \sqrt{b}\, r^2 \sin\,\theta, \tag{137}$$

where

$$\check{c} = \sqrt{b/d} = \sqrt{-g}/\sqrt{-N}\,. \tag{138}$$

Using prime ($'$) to represent $\partial/\partial r$, Ampere's law (27) and (132,133) require that

$$0 = (\sqrt{-N}N^{\dashv[01]})_{,1} = \left(\frac{\check{s}r^2 sin\,\theta}{\sqrt{d}}\right)'. \tag{139}$$

From (139,134), this means that for some constant Q we have

$$\frac{\check{s}r^2}{\sqrt{d}} = \frac{\check{s}r^2}{\sqrt{b-\check{s}^2}} = \frac{Q\sqrt{2}\, i}{\Lambda_b^{1/2}}. \tag{140}$$

Solving this for $\check{s}^2$ gives

$$\check{s}^2 = \frac{2bQ^2}{2Q^2 - \Lambda_b r^4}\,. \tag{141}$$

From (140,141) we can derive the useful relationship

$$\check{s}' = \frac{(\check{s}^2)'}{2\check{s}} = \frac{1}{2\check{s}}\left(\frac{2b'Q^2}{2Q^2-\Lambda_b r^4} + \frac{8b\Lambda_b r^3 Q^2}{2Q^2-\Lambda_b r^4}\left(\frac{\check{s}^2}{2bQ^2}\right)\right) = \frac{\check{s}}{b}\left(\frac{b'}{2} - \frac{2d}{r}\right). \tag{142}$$

The connection equations (50) are solved in [63, 64]. In terms of our variables, the non-zero connections are

$$\begin{aligned}
&\tilde\Gamma^1_{00}=\frac{a}{2}(ab)'+\frac{4a^2\check s^2}{r}\ \ ,\ \tilde\Gamma^0_{10}=\tilde\Gamma^0_{01}=\frac{(ab)'}{2ab}+\frac{2\check s^2}{br}\ \ ,\ \tilde\Gamma^1_{11}=\frac{-a'}{2a},\\
&\tilde\Gamma^2_{12}=\tilde\Gamma^2_{21}=\tilde\Gamma^3_{13}=\tilde\Gamma^3_{31}=\frac{1}{r},\\
&\tilde\Gamma^1_{22}=-ar\ \ ,\ \tilde\Gamma^1_{33}=-ar\sin^2\!\theta\ \ ,\ \tilde\Gamma^3_{23}=\tilde\Gamma^3_{32}=\cot\theta\ \ ,\ \tilde\Gamma^2_{33}=-\sin\theta\cos\theta,\\
&\tilde\Gamma^2_{02}=-\tilde\Gamma^2_{20}=\tilde\Gamma^3_{03}=-\tilde\Gamma^3_{30}=-\frac{a\check s}{r}\ \ ,\ \tilde\Gamma^1_{10}=-\tilde\Gamma^1_{01}=-\frac{2a\check s}{r},
\end{aligned}\tag{143}$$

$$\tilde\Gamma^\alpha_{\alpha 0}=0,\ \ \tilde\Gamma^\alpha_{\alpha 1}=\frac{b'}{2b}+\frac{2\check s^2}{br}+\frac{2}{r},\ \ \tilde\Gamma^\alpha_{\alpha 2}=\cot\theta,\ \ \tilde\Gamma^\alpha_{\alpha 3}=0.\tag{144}$$

The Ricci tensor is also calculated in [63, 64]. From (144) we have $\tilde\Gamma^\alpha_{\alpha[\nu,\mu]}=0$ as expected from (36), and this means that $\tilde{\mathcal R}_{\nu\mu}=\tilde R_{\nu\mu}$. In terms of our variables, and using our own sign convention, the non-zero components of the Ricci tensor are

$$\begin{aligned}
-\tilde{\mathcal R}_{00}\ &=\ -\frac{aba''}{2}-\frac{a^2b''}{2}-\frac{3aa'b'}{4}+\frac{a^2b'b'}{4b}-\frac{a}{r}(ab'+a'b)-\frac{8a^2\check s\check s'}{r}\\
&\quad+\frac{a^2\check s^2}{r}\left(\frac{3b'}{b}-\frac{3a'}{a}-\frac{10}{r}+\frac{8\check s^2}{br}\right),
\end{aligned}\tag{145}$$

$$-\tilde{\mathcal R}_{11}\ =\frac{a''}{2a}+\frac{b''}{2b}-\frac{b'b'}{4b^2}+\frac{3a'b'}{4ab}+\frac{a'}{ar}+\frac{4\check s\check s'}{br}+\frac{\check s^2}{br}\left(\frac{3a'}{a}+\frac{4\check s^2}{br}-\frac{2}{r}\right),\tag{146}$$

$$-\tilde{\mathcal R}_{22}\ =\frac{ar}{2}\left(\frac{2a'}{a}+\frac{b'}{b}\right)+a-1+\frac{2a\check s^2}{b},\tag{147}$$

$$-\tilde{\mathcal R}_{33}\ =-\tilde{\mathcal R}_{22}\sin^2\!\theta,\tag{148}$$

$$-\tilde{\mathcal R}_{[10]}=2\left(\frac{a\check s}{r}\right)'+\frac{6a\check s}{r^2}.\quad \{[63]\ \textit{has an error here}\}\tag{149}$$

From (131,136,138,148), the symmetric part of the field equations (48) is

$$0=\tilde{\mathcal R}_{00}+\Lambda_bN_{00}+\Lambda_zg_{00}=\tilde{\mathcal R}_{00}+\Lambda_bab+\Lambda_z\frac{ab}{\check c}\ ,\tag{150}$$

$$0=\tilde{\mathcal R}_{11}+\Lambda_bN_{11}+\Lambda_zg_{11}=\tilde{\mathcal R}_{11}-\Lambda_b\frac{1}{a}-\Lambda_z\frac{1}{a\check c}\ ,\tag{151}$$

$$0=\tilde{\mathcal R}_{22}+\Lambda_bN_{22}+\Lambda_zg_{22}=\tilde{\mathcal R}_{22}-\Lambda_br^2-\Lambda_z\check cr^2,\tag{152}$$

$$0=\tilde{\mathcal R}_{33}+\Lambda_bN_{33}+\Lambda_zg_{33}=(\tilde{\mathcal R}_{22}+\Lambda_bN_{22}+\Lambda_zg_{22})\sin^2\!\theta.\tag{153}$$

Forming a linear combination of (151,150) and using (146,145,142,134), we find that many of the terms cancel initially and we get,

$$0=b\left(-\tilde{\mathcal R}_{11}+\Lambda_b\frac{1}{a}+\Lambda_z\frac{1}{a\check c}\right)+\frac{1}{a^2}\left(-\tilde{\mathcal R}_{00}-\Lambda_bab-\Lambda_z\frac{ab}{\check c}\right)\tag{154}$$

$$\begin{aligned}
&=\frac{4\check s\check s'}{r}+\frac{\check s^2}{r}\left(\frac{4\check s^2}{br}-\frac{2}{r}\right)-\frac{b'}{r}-\frac{8\check s\check s'}{r}+\frac{\check s^2}{r}\left(\frac{3b'}{b}-\frac{10}{r}+\frac{8\check s^2}{br}\right)\\
&=-\frac{4\check s}{r}\left[\frac{\check s}{b}\left(\frac{b'}{2}-\frac{2d}{r}\right)\right]+\frac{12\check s^2}{r}\left(\frac{\check s^2}{br}-\frac{1}{r}\right)-\frac{b'}{r}+\frac{3\check s^2b'}{br}\\
&=-\frac{d}{br^2}\left(4\check s^2+rb'\right).
\end{aligned}\tag{155}$$

From (141) this requires

$$0=\frac{8bQ^2}{2Q^2-\Lambda_b r^4}+rb'. \tag{156}$$

Solving (156) and using (141,134,138) gives identical results to [63, 64],

$$b=1-\frac{2Q^2}{\Lambda_b r^4}, \tag{157}$$

$$\check{s}=\sqrt{\frac{2bQ^2}{2Q^2-\Lambda_b r^4}}=\frac{\sqrt{2}\,iQ}{\sqrt{\Lambda_b}\,r^2}, \tag{158}$$

$$d=b-\check{s}^2=1, \tag{159}$$

$$\check{c}=\sqrt{b/d}=\sqrt{1-\frac{2Q^2}{\Lambda_b r^4}}\,. \tag{160}$$

To find the variable "a", the 22 component of the field equations will be used. The solution is guessed to be that of [63, 64] plus an extra term $-\Lambda_z V/r$,

$$a=1-\frac{2M}{r}-\frac{\Lambda_b r^2}{3}-\frac{\Lambda_z V}{r}. \tag{161}$$

Because "b" and "$\check{s}$" are the same as [63, 64], we just need to look at the extra terms that result from Λ_z. Using (152,147,161,155,160) gives,

$$0=\,-\tilde{\mathcal{R}}_{22}+\Lambda_b r^2+\Lambda_z\check{c}r^2=\frac{ar}{2}\left(\frac{2a'}{a}+\frac{b'}{b}\right)+a-1+\frac{2a\check{s}^2}{b}+\Lambda_b r^2+\Lambda_z\check{c}r^2 \tag{162}$$

$$=\,-\Lambda_z\left[r\left(\frac{V}{r}\right)'+\frac{Vb'}{2b}+\frac{V}{r}+\frac{2V\check{s}^2}{rb}-\check{c}r^2\right]=-\Lambda_z\left[V'-r^2\check{c}\right]. \tag{163}$$

This same equation is also obtained if the 11 or 00 components of the field equations are used. The solution for $V(r)$ can be written in terms of an elliptic integral but we will not need to calculate it. With (163) and the definition

$$\hat{V}=\frac{r\Lambda_b}{Q^2}\left(V-\frac{r^3}{3}\right) \tag{164}$$

we get the following results which will be used shortly,

$$\hat{V}'=\frac{\hat{V}}{r}+\frac{r^3\Lambda_b(\check{c}-1)}{Q^2},\qquad\qquad \frac{Q^2}{\Lambda_b r}\left(\frac{\hat{V}}{r^2}\right)'=\check{c}-1-\frac{Q^2\hat{V}}{\Lambda_b r^4}. \tag{165}$$

Next we consider the antisymmetric part of the field equations (47), where only the 10 component is non-vanishing. Using (149,131,158,161) gives

$$F_{01}=\frac{\Lambda_b^{-1/2}}{\sqrt{2}\,i}(\tilde{\mathcal{R}}_{[01]}+\Lambda_b N_{[01]})=\frac{\Lambda_b^{-1/2}}{\sqrt{2}\,i}\left[2\left(\frac{a\check{s}}{r}\right)'+\frac{6a\check{s}}{r^2}+\Lambda_b\check{s}\right] \tag{166}$$

$$=2\left(\frac{aQ}{\Lambda_b r^3}\right)'+\frac{6aQ}{\Lambda_b r^4}+\frac{Q}{r^2}=\frac{Q}{r^2}\left(1+\frac{2a'}{\Lambda_b r}\right) \tag{167}$$

Using (136,157,158,159,160,161,163,167,164,165) we can put the solution in its final form. The solution is

$$ds^2=\check{c}adt^2-\frac{1}{\check{c}a}\,dr^2-\check{c}r^2d\theta^2-\check{c}r^2sin^2\theta d\phi^2, \tag{168}$$

$$f^{10}=\frac{Q}{\check{c}r^2},\qquad \sqrt{-N}=r^2sin\,\theta,\qquad \sqrt{-g}=\check{c}r^2sin\,\theta, \tag{169}$$

$$F_{01} = -A_0' = \frac{Q}{r^2}\left[1+\frac{4M}{\Lambda_b r^3}-\frac{4\Lambda}{3\Lambda_b}+2\left(\check{c}-1-\frac{Q^2\hat{V}}{\Lambda_b r^4}\right)\left(1-\frac{\Lambda}{\Lambda_b}\right)\right], \tag{170}$$

$$a = 1-\frac{2M}{r}-\frac{\Lambda r^2}{3}+\frac{Q^2\hat{V}}{r^2}\left(1-\frac{\Lambda}{\Lambda_b}\right), \tag{171}$$

where prime ($'$) means $\partial/\partial r$, and $\check{c}$ and $\hat{V}$ are very close to one for ordinary radii,

$$\check{c} = \sqrt{1-\frac{2Q^2}{\Lambda_b r^4}} = 1-\frac{Q^2}{\Lambda_b r^4}\cdots-\frac{(2i)!}{[i!]^2 4^i(2i-1)}\left(\frac{2Q^2}{\Lambda_b r^4}\right)^i, \tag{172}$$

$$\hat{V} = \frac{r\Lambda_b}{Q^2}\left(\int r^2\check{c}\,dr-\frac{r^3}{3}\right) = 1+\frac{Q^2}{10\Lambda_b r^4}\cdots+\frac{(2i)!}{i!(i+1)!\,4^i(4i+1)}\left(\frac{2Q^2}{\Lambda_b r^4}\right)^i. \tag{173}$$

With $\Lambda_z=0, \Lambda_b=\Lambda$ we get the Papapetrou solution[63, 64] of the unmodified Einstein-Schrödinger theory. In this case the $M/\Lambda_b r^3$ term in (170) would be huge from (16), and the Q^2/r^2 term in (171) disappears, which is why the Papapetrou solution was found to be unsatisfactory in [63]. However, we are instead assuming $\Lambda_b \approx -\Lambda_z$ from (14,15). In this case the solution matches the Reissner-Nordström solution except for terms which are negligible for ordinary radii. To see this, first recall that $\Lambda/\Lambda_b \sim 10^{-122}$ from (16), so the Λ terms are all completely negligible. Ignoring the Λ terms and keeping only the leading order terms in (170,171,172) gives

$$F_{01} = \frac{Q}{r^2}\left[1+\frac{4M}{\Lambda_b r^3}-\frac{4Q^2}{\Lambda_b r^4}\right]+\mathcal{O}(\Lambda_b^{-2}), \tag{174}$$

$$A_0 = \frac{Q}{r}\left[1+\frac{M}{\Lambda_b r^3}-\frac{4Q^2}{5\Lambda_b r^4}\right]+\mathcal{O}(\Lambda_b^{-2}), \tag{175}$$

$$a = 1-\frac{2M}{r}+\frac{Q^2}{r^2}\left[1+\frac{Q^2}{10\Lambda_b r^4}\right]+\mathcal{O}(\Lambda_b^{-2}), \tag{176}$$

$$\check{c} = 1-\frac{Q^2}{\Lambda_b r^4}+\mathcal{O}(\Lambda_b^{-2}). \tag{177}$$

For the smallest radii probed by high-energy particle physics we get from (63),

$$\frac{Q^2}{\Lambda_b r^4} \sim 10^{-66}. \tag{178}$$

The worst-case value of $M/\Lambda_b r^3$ might be near the Schwarzschild radius r_s of black holes where $r=r_s=2M$ and $M/\Lambda_b r^3 = 1/2\Lambda_b r_s^2$. This value will be largest for the lightest black holes, and the lightest black hole that we can expect to observe would be of about one solar mass, where we have

$$\frac{M}{\Lambda_b r^3} \sim \frac{1}{2\Lambda_b r_s^2} = \frac{1}{2\Lambda_b}\left(\frac{c^2}{2Gm_\odot}\right)^2 \sim 10^{-77}. \tag{179}$$

Also, an electron has $M = Gm_e/c^2 = 7\times10^{-56}cm$, and using (14) and the smallest radii probed by high-energy particle physics ($10^{-17}cm$) we have

$$\frac{M}{\Lambda_b r^3} \sim \frac{7\times10^{-56}}{10^{66}(10^{-17})^3} \sim 10^{-70}. \tag{180}$$

From (178,179,180,16) we see that our electric monopole solution (168-171) has a fractional difference from the Reissner-Nordström solution of at most 10^{-66} for worst-case radii accessible to measurement. Clearly our solution does not have the deficiencies of the Papapetrou solution[63, 64] in the original theory, and it is almost

certainly indistinguishable from the Reissner-Nordström solution experimentally. Also, when this solution is expressed in Newman-Penrose tetrad form, it can be shown to be of Petrov Type-D[21]. And of course the solution reduces to the Schwarzschild solution for $Q=0$. And we see that the solution goes to the Reissner-Nordström solution exactly in the limit as $\Lambda_b\rightarrow\infty$.

The only significant difference between our electric monopole solution and the Reissner-Nordström solution occurs on the Planck scale. From (168,172), the surface area of the solution is[49],

$$\begin{pmatrix}\text{surface}\\ \text{area}\end{pmatrix}=\int_0^\pi d\theta\int_0^{2\pi}d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}}=4\pi r^2\check{c}=4\pi r^2\sqrt{1-\frac{2Q^2}{\Lambda_b r^4}}. \tag{181}$$

The origin of the solution is where the surface area vanishes, so in our coordinates the origin is not at $r=0$ but rather at

$$r_0=\sqrt{Q}(2/\Lambda_b)^{1/4}. \tag{182}$$

From (62,14) we have $r_0\sim l_P\sim 10^{-33}cm$ for an elementary charge, and $r_0\ll 2M$ for any realistic astrophysical black hole. For $Q/M<1$ the behavior at the origin is hidden behind an event horizon nearly identical to that of the Reissner-Nordström solution. For $Q/M>1$ where there is no event horizon, the behavior at the origin differs markedly from the simple naked singularity of the Reissner-Nordström solution. For the Reissner-Nordström solution all of the relevant fields have singularities at the origin, $g_{00}\sim Q^2/r^2$, $A_0=Q/r$, $F_{01}=Q/r^2$, $R_{00}\sim 2Q^4/r^6$ and $R_{11}\sim 2/r^2$. For our solution the metric has a less severe singularity at the origin, $g_{11}\sim-\sqrt{r_0}/2\sqrt{r-r_0}$ and $\sqrt{-g}=0$. Also, the fields $N_{\mu\nu}$, $N^{\dashv\nu\mu}$, $\sqrt{-N}$, A_ν, $\sqrt{-g}f^{\nu\mu}$, $\sqrt{-g}f_{\nu\mu}$, $\sqrt{-g}g^{\nu\mu}$, $\sqrt{-g}g_{\nu\mu}$, and the functions "a" and $\hat{V}$ all have finite nonzero values and derivatives at the origin, because it can be shown that $\hat{V}(r_0)=\sqrt{2}\,[\Gamma(1/4)]^2/6\sqrt{\pi}-2/3=1.08137$. The fields $F_{\nu\mu}$, $\tilde{\Gamma}^\alpha_{\mu\nu}$ and $\sqrt{-g}\,\tilde{\mathcal{R}}_{\nu\mu}$ are also finite and nonzero at the origin, so if we use the tensor density form of the field equations (65,27), there is no ambiguity as to whether the field equations are satisfied at this location.

Finally let us consider the result from (63) that $|f^\mu{}_\sigma\Lambda_b^{-1/2}|<10^{-33}$ for worst-case electromagnetic fields accessible to measurement. The "smallness" of this value may seem unappealing at first, considering that $g^{\mu\nu}$ and $f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ are part of the total field $(\sqrt{-N}/\sqrt{-g})N^{\dashv\nu\mu}=g^{\mu\nu}+f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ as in (25). However, for an elementary charge, $|f^{\mu\nu}\Lambda_b^{-1/2}|$ is not really small if one compares it to $g^{\mu\nu}-\eta^{\mu\nu}$ instead of $g^{\mu\nu}$. Our electric monopole solution (168,169,171) has $g^{00}\approx 1+2M/r+Q^2/r^2$ and $f^{01}\approx Q/r^2$. So for an elementary charge, we see from (62,14) that $|f^{01}\Lambda_b^{-1/2}|\sim Q^2/r^2$ for any radius.

8. Conclusions

The Einstein-Schrödinger theory was modified to account for a quantum-mechanical effect. The Einstein equations of this theory have zero divergence with the Christoffel connection, allowing additional (non-electromagnetic) fields to be included in the theory. The field equations match the ordinary electro-vac Einstein and Maxwell equations except for additional terms which are $<10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. The theory avoids ghosts in the sense that ghosts could only exist if their frequency exceeds the cutoff frequency $\omega_c\sim 1/l_P$ of zero-point fluctuations. The Einstein-Infeld-Hoffmann (EIH) equations of motion for this theory match the equations of motion for Einstein-Maxwell

theory to Newtonian/Coulombian order, which proves the existence of a Lorentz force. An exact electric monopole solution exists for this theory, and it matches the Reissner-Nordström solution except for additional terms which are $\sim 10^{-66}$ of the usual terms for worst-case radii accessible to measurement. The theory becomes exactly electro-vac Einstein-Maxwell theory in the limit as $|\Lambda_z|\to\infty$, $\Lambda_b\to\infty$, or more precisely as $\omega_c\to\infty$. It seems unlikely that there is a test which is sensitive enough to discriminate the theory from electro-vac Einstein-Maxwell theory.

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Appendix A. Extraction of a connection addition from the Hermitianized Ricci tensor

Substituting $\tilde\Gamma^\alpha_{\nu\mu}=\Gamma^\alpha_{\nu\mu}+\Upsilon^\alpha_{\nu\mu}$ from (59,22) into (12) gives

$$\begin{aligned}
\mathcal{R}_{\nu\mu}(\tilde\Gamma) &= 2[(\Gamma^\alpha_{\nu[\mu}+\Upsilon^\alpha_{\nu[\mu}),_{\alpha]} + (\Gamma^\sigma_{\nu[\mu}+\Upsilon^\sigma_{\nu[\mu})(\Gamma^\alpha_{\sigma|\alpha]}+\Upsilon^\alpha_{\sigma|\alpha]})] + (\Gamma^\alpha_{\alpha[\nu}+\Upsilon^\alpha_{\alpha[\nu}),_{\mu]} \quad (A.1)\\
&= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{\nu\mu,\alpha} - \Gamma^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} + \Gamma^\alpha_{\sigma\alpha}\Upsilon^\sigma_{\nu\mu} - \Gamma^\alpha_{\sigma\mu}\Upsilon^\sigma_{\nu\alpha}\\
&\qquad - \Upsilon^\alpha_{\alpha(\nu,\mu)} + \Gamma^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha} - \Upsilon^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} + \Upsilon^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha} \quad (A.2)\\
&= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{\nu\mu;\alpha} - \Upsilon^\alpha_{\alpha(\nu;\mu)} - \Upsilon^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} + \Upsilon^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha}, \quad (A.3)
\end{aligned}$$

$$R_{(\nu\mu)}(\tilde\Gamma) = R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{(\nu\mu);\alpha} - \Upsilon^\alpha_{\alpha(\nu;\mu)} - \Upsilon^\sigma_{(\nu\alpha)}\Upsilon^\alpha_{(\sigma\mu)} - \Upsilon^\sigma_{[\nu\alpha]}\Upsilon^\alpha_{[\sigma\mu]} + \Upsilon^\sigma_{(\nu\mu)}\Upsilon^\alpha_{\sigma\alpha}, \quad (A.4)$$

$$R_{[\nu\mu]}(\tilde\Gamma) = \Upsilon^\alpha_{[\nu\mu];\alpha} - \Upsilon^\sigma_{(\nu\alpha)}\Upsilon^\alpha_{[\sigma\mu]} - \Upsilon^\sigma_{[\nu\alpha]}\Upsilon^\alpha_{(\sigma\mu)} + \Upsilon^\sigma_{[\nu\mu]}\Upsilon^\alpha_{\sigma\alpha}. \quad (A.5)$$

Also, substituting $\widehat\Gamma^\alpha_{\nu\mu} = \tilde\Gamma^\alpha_{\nu\mu} + [\,\delta^\alpha_\mu A_\nu - \delta^\alpha_\nu A_\mu]\sqrt2\, i\Lambda_b^{1/2}$ from (8) into (7) and using $\tilde\Gamma^\alpha_{\nu\alpha} = \widehat\Gamma^\alpha_{(\nu\alpha)} = \tilde\Gamma^\alpha_{\alpha\nu}$ gives

$$\begin{aligned}
\mathcal{R}_{\nu\mu}(\widehat\Gamma) &= \tilde\Gamma^\alpha_{\nu\mu,\alpha} + [\,\delta^\alpha_\mu A_\nu - \delta^\alpha_\nu A_\mu],_\alpha\sqrt2\, i\Lambda_b^{1/2} - \tilde\Gamma^\alpha_{\alpha(\nu,\mu)}\\
&\quad + \left(\tilde\Gamma^\sigma_{\nu\mu} + [\,\delta^\sigma_\mu A_\nu - \delta^\sigma_\nu A_\mu]\sqrt2\, i\Lambda_b^{1/2}\right)\tilde\Gamma^\alpha_{\sigma\alpha}\\
&\quad - \left(\tilde\Gamma^\sigma_{\nu\alpha} + [\,\delta^\sigma_\alpha A_\nu - \delta^\sigma_\nu A_\alpha]\sqrt2\, i\Lambda_b^{1/2}\right)\left(\tilde\Gamma^\alpha_{\sigma\mu} + [\,\delta^\alpha_\mu A_\sigma - \delta^\alpha_\sigma A_\mu]\sqrt2\, i\Lambda_b^{1/2}\right)\\
&\quad + 2(n-1)A_\nu A_\mu\Lambda_b \quad (A.6)\\
&= \tilde\Gamma^\alpha_{\nu\mu,\alpha} + 2A_{[\nu,\mu]}\sqrt2\, i\Lambda_b^{1/2} - \tilde\Gamma^\alpha_{\alpha(\nu,\mu)} + \tilde\Gamma^\sigma_{\nu\mu}\tilde\Gamma^\alpha_{\sigma\alpha} + [\,A_\nu\tilde\Gamma^\alpha_{\mu\alpha} - A_\mu\tilde\Gamma^\alpha_{\nu\alpha}]\sqrt2\, i\Lambda_b^{1/2}\\
&\quad - \tilde\Gamma^\sigma_{\nu\alpha}\tilde\Gamma^\alpha_{\sigma\mu} - [\,\tilde\Gamma^\sigma_{\nu\mu}A_\sigma - \tilde\Gamma^\sigma_{\nu\sigma}A_\mu]\sqrt2\, i\Lambda_b^{1/2} - [\,A_\nu\tilde\Gamma^\alpha_{\alpha\mu} - A_\alpha\tilde\Gamma^\alpha_{\nu\mu}]\sqrt2\, i\Lambda_b^{1/2}\\
&\quad + 2A_\nu A_\mu(1-n-1+1)\Lambda_b + 2(n-1)A_\nu A_\mu\Lambda_b \quad (A.7)\\
&= \tilde\Gamma^\alpha_{\nu\mu,\alpha} - \tilde\Gamma^\alpha_{\alpha(\nu,\mu)} + \tilde\Gamma^\sigma_{\nu\mu}\tilde\Gamma^\alpha_{\sigma\alpha} - \tilde\Gamma^\sigma_{\nu\alpha}\tilde\Gamma^\alpha_{\sigma\mu} + 2A_{[\nu,\mu]}\sqrt2\, i\Lambda_b^{1/2} \quad (A.8)\\
&= \mathcal{R}_{\nu\mu}(\tilde\Gamma) + 2A_{[\nu,\mu]}\sqrt2\, i\Lambda_b^{1/2}. \quad (A.9)
\end{aligned}$$

Appendix B. A divergence identity

Using only the definitions (5,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$, and the identity (34) gives,

$$\left(N^{(\mu}{}_{\nu)} - \frac12\delta^\mu_\nu N^\rho_\rho\right)_{;\,\mu} - \frac32 f^{\sigma\rho}N_{[\sigma\rho,\nu]}\sqrt2\, i\Lambda_b^{-1/2} \quad (B.1)$$

$$= \frac{1}{2}g^{\sigma\rho}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})-\frac{1}{2}f^{\sigma\rho}(N_{[\sigma\rho];\nu}+N_{[\rho\nu];\sigma}+N_{[\nu\sigma];\rho})\sqrt{2}\,i\Lambda_b^{-1/2} \tag{B.2}$$

$$= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[N^{\dashv(\sigma\rho)}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})+N^{\dashv[\sigma\rho]}(N_{[\sigma\rho];\nu}+N_{[\rho\nu];\sigma}+N_{[\nu\sigma];\rho})\right] \tag{B.3}$$

$$= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[\,N^{\dashv\sigma\rho}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})+N^{\dashv\sigma\rho}(N_{[\rho\nu];\sigma}+N_{[\nu\sigma];\rho}-N_{[\rho\sigma];\nu})\right] \tag{B.4}$$

$$= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}(N_{\rho\nu;\sigma}+N_{\nu\sigma;\rho}-N_{\rho\sigma;\nu}) \tag{B.5}$$

$$= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[N^{\dashv\sigma\rho}(N_{\rho\nu;\sigma}+N_{\nu\sigma;\rho})-N^{\dashv\sigma\rho}(N_{\rho\sigma,\nu}-\Gamma^\alpha_{\rho\nu}N_{\alpha\sigma}-\Gamma^\alpha_{\sigma\nu}N_{\rho\alpha})\right] \tag{B.6}$$

$$= -\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}(N^{\dashv\sigma\rho}{}_{;\sigma}N_{\rho\nu}+N^{\dashv\sigma\rho}{}_{;\rho}N_{\nu\sigma})-\frac{1}{\sqrt{-g}}(\sqrt{-N}\,)_{;\nu} \tag{B.7}$$

$$= -\frac{1}{2}\left[\left(\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}\right)_{;\sigma}N_{\rho\nu}+\left(\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}\right)_{;\rho}N_{\nu\sigma}\right] \tag{B.8}$$

$$= -\frac{1}{2}\left[(g^{\rho\sigma}+f^{\rho\sigma}\sqrt{2}\,i\Lambda_b^{-1/2})_{;\sigma}N_{\rho\nu}+(g^{\rho\sigma}+f^{\rho\sigma}\sqrt{2}\,i\Lambda_b^{-1/2})_{;\rho}N_{\nu\sigma}\right] \tag{B.9}$$

$$= f^{\sigma\rho}{}_{;\sigma}N_{[\rho\nu]}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{B.10}$$

Appendix C. Variational derivatives for fields with the symmetry $\tilde\Gamma^\sigma_{[\mu\sigma]}=0$

The field equations associated with a field with symmetry properties must have the same number of independent components as the field. For a field with the symmetry $\tilde\Gamma^\sigma_{[\mu\sigma]}=0$, the field equations can be found by introducing a Lagrange multiplier Ω^μ,

$$0=\delta\int(\mathcal{L}+\Omega^\mu\tilde\Gamma^\sigma_{[\mu\sigma]})d^nx. \tag{C.1}$$

Minimizing the integral with respect to Ω^μ shows that the symmetry is enforced. Using the definition,

$$\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}=\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho}}-\left(\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho,\omega}}\right)_{,\omega}\cdots, \tag{C.2}$$

and minimizing the integral with respect to $\tilde\Gamma^\beta_{\tau\rho}$ gives

$$0=\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}+\Omega^\mu\delta^\sigma_\beta\delta^\tau_{[\mu}\delta^\rho_{\sigma]}=\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}+\frac{1}{2}(\Omega^\tau\delta^\rho_\beta-\delta^\tau_\beta\Omega^\rho). \tag{C.3}$$

Contracting this on the left and right gives

$$\Omega^\rho=\frac{2}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}}=-\frac{2}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\rho\alpha}}. \tag{C.4}$$

Substituting (C.4) back into (C.3) gives

$$0=\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}-\frac{\delta^\tau_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}}-\frac{\delta^\rho_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\tau\alpha}}. \tag{C.5}$$

In (C.4,C.5) the index contractions occur after the derivatives. Contracting (C.5) on the right and left gives the same result, so it has the same number of independent components as $\tilde\Gamma^\alpha_{\mu\nu}$. This is a general expression for the field equations associated with a field having the symmetry $\tilde\Gamma^\sigma_{[\mu\sigma]}=0$.

Appendix D. Solution for $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$

Here we invert the definitions (5,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$ to obtain (57,58), the approximation of $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. First let us define the notation

$$\hat{f}^{\nu\mu} = f^{\nu\mu}\sqrt{2}\, i\, \Lambda_b^{-1/2}. \tag{D.1}$$

We assume that $|\hat{f}^{\nu}{}_{\mu}| \ll 1$ for all components of the unitless field $\hat{f}^{\nu}{}_{\mu}$, and find a solution in the form of a power series expansion in $\hat{f}^{\nu}{}_{\mu}$. Lowering an index on the equation $(\sqrt{-N}/\sqrt{-g}\,)N^{\dashv\mu\nu} = g^{\nu\mu} + \hat{f}^{\nu\mu}$ from (5,24) gives

$$\frac{\sqrt{-N}}{\sqrt{-g}} N^{\dashv\mu}{}_{\alpha} = \delta^{\mu}_{\alpha} - \hat{f}^{\mu}{}_{\alpha}. \tag{D.2}$$

Let us consider the tensor $\hat{f}^{\mu}{}_{\alpha} = \hat{f}^{\mu\nu} g_{\nu\alpha}$. Because $g_{\nu\alpha}$ is symmetric and $\hat{f}^{\mu\nu}$ is antisymmetric, it is clear that $\hat{f}^{\alpha}{}_{\alpha} = 0$. Also because $\hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu}$ is symmetric it is clear that $\hat{f}^{\nu}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu}\hat{f}^{\mu}{}_{\nu} = 0$. In matrix language therefore $tr(\hat{f}) = 0$, $tr(\hat{f}^3) = 0$, and in fact $tr(\hat{f}^p) = 0$ for any odd p. Using the well known formula $det(e^M) = exp\,(tr(M))$ and the power series $ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4\ldots$ we then get[58],

$$ln(det(I-\hat{f})) = tr(ln(I-\hat{f})) = -\frac{1}{2}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho} + (\hat{f}^4)\ldots \tag{D.3}$$

Here the notation $(\hat{f}^4)$ refers to terms like $\hat{f}^{\tau}{}_{\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}\hat{f}^{\rho}{}_{\tau}$. Taking $ln(det())$ on both sides of (D.2) using the result (D.3) and the identities $det(sM) = s^n det(M)$ and $det(M^{-1}) = 1/det(M)$ gives

$$ln\left(\frac{\sqrt{-N}}{\sqrt{-g}}\right) = \frac{1}{(n-2)}\, ln\left(\frac{N^{(n/2-1)}}{g^{(n/2-1)}}\right) = -\frac{1}{2(n-2)}\,\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho} + (\hat{f}^4)\ldots \tag{D.4}$$

Taking e^x on both sides of (D.4) and using $e^x = 1 + x + x^2/2\ldots$ gives

$$\frac{\sqrt{-N}}{\sqrt{-g}} = 1 - \frac{1}{2(n-2)}\,\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho} + (\hat{f}^4)\ldots \tag{D.5}$$

Using the power series $(1-x)^{-1} = 1 + x + x^2 + x^3\ldots$, or multiplying (D.2) term by term, we can calculate the inverse of (D.2) to get[58]

$$\frac{\sqrt{-g}}{\sqrt{-N}} N^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \hat{f}^{\nu}{}_{\mu} + \hat{f}^{\nu}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + \hat{f}^{\nu}{}_{\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^4)\ldots \tag{D.6}$$

$$N_{\nu\mu} = \frac{\sqrt{-N}}{\sqrt{-g}}(g_{\nu\mu} + \hat{f}_{\nu\mu} + \hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu} + \hat{f}_{\nu\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^4)\ldots). \tag{D.7}$$

Here the notation $(\hat{f}^4)$ refers to terms like $\hat{f}_{\nu\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}\hat{f}^{\rho}{}_{\mu}$. Since $\hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu}$ is symmetric and $\hat{f}_{\nu\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu}$ is antisymmetric, we obtain from (D.7,D.5,D.1) the final result (57,58).

Appendix E. Derivation of the Einstein-Schrödinger theory from a purely affine Lagrangian density

If the theory proposed in this paper is correct, we might expect that it can be derived from some kind of simple principles. Here we will show that the original unmodified Einstein-Schrödinger theory can be derived from a Lagrangian density $\mathcal{L}(\widehat{\Gamma})$ which

depends only on an affinity $\widehat{\Gamma}^\alpha_{\sigma\mu}$ with no symmetry properties, resulting in the field equations

$$0=\delta S,\qquad S=\int\mathcal{L}(\widehat{\Gamma})dx^1dx^2...dx^n, \tag{E.1}$$

$$\Rightarrow\quad 0=\frac{\delta\mathcal{L}}{\delta\widehat{\Gamma}^\beta_{\tau\rho}}=\frac{\partial\mathcal{L}}{\partial\widehat{\Gamma}^\beta_{\tau\rho}}-\left(\frac{\partial\mathcal{L}}{\partial\widehat{\Gamma}^\beta_{\tau\rho,\omega}}\right)_{,\,\omega}+\left(\frac{\partial\mathcal{L}}{\partial\widehat{\Gamma}^\beta_{\tau\rho,\omega,\nu}}\right)_{,\,\omega,\,\nu}..., \tag{E.2}$$

where the field equations require

$$\mathcal{L}_{,\beta}-\widehat{\Gamma}^\alpha_{(\alpha\beta)}\mathcal{L}=0. \tag{E.3}$$

Equation (E.3) is a simple generalization of the result $\mathcal{L}_{,\beta}-\Gamma^\alpha_{\alpha\beta}\mathcal{L}=0$ that occurs with ordinary vacuum general relativity. We will also show that the Einstein-Schrödinger theory appears to be unique in that it can be derived from a Lagrangian density which satisfies (E.3).

Suppose we view (E.2,E.3) as requirements for a purely classical field theory. The task is then to solve (E.2,E.3) for the unknowns $\mathcal{L}(\widehat{\Gamma})$ and $\widehat{\Gamma}^\alpha_{\mu\nu}$. For an arbitrary $\mathcal{L}(\widehat{\Gamma})$, these equations constitute more equations than unknowns, and no nontrivial solution for $\widehat{\Gamma}^\alpha_{\mu\nu}$ can be expected. However, for the correct $\mathcal{L}(\widehat{\Gamma})$, (E.3) is contained in (E.2) and nontrivial solutions can be expected. A Lagrangian density which allows a solution to (E.2,E.3) is the following,

$$\mathcal{L}(\widehat{\Gamma})=\frac{\Lambda_b}{16\pi}\sqrt{-det(N_{\nu\mu})}, \tag{E.4}$$

where $N_{\nu\mu}$ is simply defined to be

$$N_{\nu\mu}=-\hat{\mathcal{R}}_{\nu\mu}/\Lambda_b=-(\tilde{\mathcal{R}}_{\nu\mu}+2A_{[\nu,\mu]}\sqrt{2}\,i\Lambda_b^{1/2})/\Lambda_b, \tag{E.5}$$

and $\hat{\mathcal{R}}_{\sigma\mu}=\mathcal{R}_{\sigma\mu}(\widehat{\Gamma})$ is the Hermitianized Ricci tensor from (7). Here we have decomposed $\widehat{\Gamma}^\alpha_{\nu\mu}$ into $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_σ as in (6,8,9), and we have also used (A.8) and $\tilde{\mathcal{R}}_{\nu\mu}=\mathcal{R}_{\nu\mu}(\tilde{\Gamma})$ from (12). The connection $\tilde{\Gamma}^\alpha_{\nu\mu}$ has the symmetry (51) so it has only n^3-n independent components. From (8,51), $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_ν fully parameterize $\widehat{\Gamma}^\alpha_{\nu\mu}$ and can be treated as independent variables. From the invariance properties (18,19) of the Hermitianized Ricci tensor (7), the Lagrangian density (E.4) is real, and it is also invariant under both charge conjugation (20) and under an electromagnetic gauge transformation (21).

Now, it is simple to show that setting $\delta\mathcal{L}/\delta A_\nu=0$ and $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu}=0$ gives identical equations as in §3 except that $\Lambda_z=0$. In addition, the definition (E.5) matches (46), so that this equation and all of the subsequent equations in §4 and §5 are identical except that $\Lambda_z=0$. Therefore, the purely affine Lagrangian density (E.4,E.5) gives the same theory as the Palatini Lagrangian density (3) with $\Lambda_z=0$, which is the original Einstein-Schrödinger theory. In particular (35) is valid, and this together with (51,E.4) gives (E.3).

The derivation of the Einstein-Schrödinger theory in this manner is remarkable because the only fundamental field assumed *a priori* was the connection $\widehat{\Gamma}^\alpha_{\sigma\mu}$. The fundamental tensor $N_{\sigma\mu}$, the metric $g_{\sigma\mu}$, the field $f_{\sigma\mu}$, the electromagnetic potential A_σ, and the contraction-symmetric connection $\tilde{\Gamma}^\alpha_{\sigma\mu}$ all just appeared as convenient variables to work with when solving the field equations. We should emphasize that the same field equations also result from setting $\delta\mathcal{L}/\delta\widehat{\Gamma}^\alpha_{\sigma\mu}=0$. When this is done, one obtains a rather complicated set of field equations in the unknowns $\widehat{\Gamma}^\alpha_{\sigma\mu}$.

However, when the equations are rewritten in terms of the variable $\tilde\Gamma^\alpha_{\sigma\mu}$ from (9), much simplification occurs and one eventually obtains the ordinary Einstein-Schrödinger field equations. The same thing occurs if one sets $\delta\mathcal{L}_S/\delta\widehat\Gamma^\alpha_{\sigma\mu}=0$ using Schrödinger's[6] Lagrangian density $\mathcal{L}_S=\sqrt{-det(-R_{\sigma\mu}(\widehat\Gamma))}$, but then the equations simplify when rewritten in terms of the variable $\tilde\Gamma^\alpha_{\sigma\mu}=\widehat\Gamma^\alpha_{\sigma\mu}-2\delta^\alpha_\sigma\widehat\Gamma^\nu_{[\nu\mu]}/(n-1)$ instead of $\tilde\Gamma^\alpha_{\sigma\mu}$.

It is important to note that this simple derivation only works for Schrödinger's generalization of Einstein's theory which includes an intrinsic cosmological constant, because if $\Lambda_b=0$, the definition (E.5) would not make sense. Also note that the only reason we do not set $\Lambda_b=1$ is because we are assuming the convention that $N_{\sigma\mu}$ has values close to 1. If we chose to we would be free to absorb Λ_b into $N_{\sigma\mu}$ because both $\tilde\Gamma^\alpha_{\sigma\mu}(N_{..})$ and $R_{\sigma\mu}(\tilde\Gamma(N_{..}))$ are independent of a constant multiplier on $N_{\sigma\mu}$. We would also be free to absorb Λ_b into the definition of A_σ. Therefore, Λ_b does not need to be in either the field equations or the Lagrangian density. It is only there to make the definitions of $N_{\sigma\mu}$ and A_σ conform to conventions. The cosmological constant term has often been referred to as an undesirable complication, attached to otherwise elegant field equations to make them conform to reality. From the standpoint of the derivation above, it is nothing of the sort. Instead, Λ_b appears as the magnitude of the fundamental tensor $N_{\sigma\mu}$ when $N_{\sigma\mu}$ is put in more natural units. The cosmological constant term is not an added-on appendage to this theory but is instead an inherent part of it.

Given that the Einstein-Schrödinger theory can be derived from a Lagrangian density which obeys the equation $\mathcal{L}_{,\beta}-\widehat\Gamma^\alpha_{(\alpha\beta)}\mathcal{L}=0$ as in (E.3), we must next ask whether it is unique in this regard. While a rigorous proof is probably not possible, a strong argument will be presented below that the theory is unique in this property. With no metric to use, the forms that a scalar density can take are limited. Also, because (E.3) exists for any dimension, we must only consider forms which exist for any dimension. To discuss this topic, it is convenient to use the fields $\tilde\Gamma^\alpha_{\sigma\mu}, A_\sigma$ as defined by (9,6) instead of $\widehat\Gamma^\alpha_{\sigma\mu}$. The simplest form to consider is $\mathcal{L}=\sqrt{-N}$, where $N_{\sigma\mu}$ is a linear combination of the terms $\tilde{\mathcal{R}}_{\sigma\mu}$, $\tilde{\mathcal{R}}_{\mu\sigma}$, $\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}$, $A_{[\sigma,\mu]}$, $A_{\sigma,\mu}–\tilde\Gamma^\alpha_{\sigma\mu}A_\alpha$, $\tilde\Gamma^\alpha_{[\sigma\mu]}A_\alpha$, and $A_\sigma A_\mu$. Many other terms can be decomposed into these, such as $R_{\sigma\mu}(\tilde\Gamma^T)=\tilde{\mathcal{R}}_{\mu\sigma}+2\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}$, $\tilde{\mathcal{R}}^\alpha{}_{\alpha\sigma\mu}=2\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}$, and anything dependent on $\widehat\Gamma^\alpha_{\sigma\mu}$. Our Lagrangian density (E.4) is a special case of this form. In fact, it happens that (E.3) is satisfied for any $\mathcal{L}=\sqrt{-N}$ where $N_{\sigma\mu}=a\tilde{\mathcal{R}}_{\sigma\mu}+bA_{[\sigma,\mu]}+c\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}$ and $a\neq 0, b\neq 0$. This would initially seem to indicate that the Einstein-Schrödinger theory is not unique, except for the surprising fact that the same field equations result regardless of the coefficients in the linear combination. The $\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}$ term causes $\delta^\rho_\beta(\sqrt{-N}\,N^{\dashv[\tau\omega]})_{,\omega}$ terms in the $\delta\mathcal{L}/\delta\tilde\Gamma^\beta_{\tau\rho}=0$ field equations (50), but these are required to vanish by the $\delta\mathcal{L}/\delta A_\tau=0$ field equations (27). Also, (E.3) requires that $\tilde\Gamma^\alpha_{\alpha[\mu,\sigma]}=(ln\mathcal{L})_{,[\mu,\sigma]}=0$ from (36), so this term is of no consequence. Different field equations result if any other terms are included in $N_{\sigma\mu}$, but then (E.3) is no longer satisfied. To argue the case for uniqueness, we must next consider more complicated forms. The most obvious generalization of a single $\sqrt{-N}$ consists of linear combinations of such terms, $\sqrt{-{}^1N}$ and $\sqrt{-{}^2N}$ etcetera. The resulting field equations contain different $N^{\dashv\sigma\mu}$ terms, and there is just no way to contract the equations to remove these terms as we did in (35). Linear combinations of terms such as $\sqrt{-{}^1N}\sqrt{-{}^2N}/\sqrt{-{}^3N}$ have the same characteristic. Next one can include linear combinations of terms like $\sqrt{-{}^1N}\,{}^1N^{\dashv\sigma\mu}\,{}^2N_{\mu\sigma}$. In this case the field

equations contain terms with different powers of ${}^1N^{\dashv\sigma\mu}$. From trying a few of these, it seems very likely that the simplicity of (E.3) demands simplicity in the Lagrangian density, and that the only real prospect is a single $\sqrt{-N}$ as we considered originally.

At first glance, these results might seem unimportant because the original Einstein-Schrödinger theory does not seem to represent anything physical. However, the theory proposed in this paper is just the Einstein-Schrödinger theory with a quantization effect, namely a Λ_z term caused by zero-point fluctuations. A spin-1/2 $\mathcal{L}_m$ term can be viewed as another quantization effect, namely the first quantization of our charged monopole solution. And as shown in [22], when these two terms are included in the Lagrangian we get an extremely close approximation to one-particle quantum electrodynamics. From this perspective, the fact that the original Einstein-Schrödinger theory can be derived from simple principles may be important because this theory is the purely classical core of a theory which represents a large part of reality. Furthermore, if one was to try to second quantize the theory, the most obvious approach would be to use path integral methods with the action (E.1,E.4,E.5). Since we are proposing that spin-1/2 particles have their origin as singular solutions of the field equations, both the Λ_z and the spin-1/2 part of our theory might be expected to appear as quantization effects using the purely classical action (E.1,E.4,E.5), and adding up the $e^{iS/\hbar}$ amplitudes for all "paths" of the field $\widehat{\Gamma}^\alpha_{\mu\nu}$. Now it is unclear whether such a quantization scheme would work, or how practical it would be in terms of being able to do the calculations and predict experimental results. However, it is at least theoretically possible.

Appendix F. Verification that the EIH method applied to Einstein-Maxwell theory gives the equations of motion of the Darwin Lagrangian to Post-Coulombian order

Here we will compare the post-Coulombian equations of motion for Einstein-Maxwell theory obtained by two authors[24, 61] using the EIH method, to the equations of motion obtained from the Darwin Lagrangian[55]. For two particles the Darwin Lagrangian takes the form

$$L_a = \frac{m_a v_a^2}{2} + \frac{1}{8}\frac{m_a v_a^4}{c^2} - e_a\frac{e_b}{R_{ab}} + \frac{e_a}{2c^2}\frac{e_b}{R_{ab}}\left[\mathbf{v}_a\cdot\mathbf{v}_b + (\mathbf{v}_a\cdot\mathbf{n}_{ab})(\mathbf{v}_b\cdot\mathbf{n}_{ab})\right]. \tag{F.1}$$

Here we are using the notation

$$\dot r^i_a = v^i_a,\quad \dot r^i_b = v^i_b,\quad r^i_{ab} = r^i_a - r^i_b,\quad v^i_{ab} = v^i_a - v^i_b,\quad n^i_{ab} = r^i_{ab}/R_{ab},\quad R^2_{ab} = r^i_{ab}r^i_{ab}. \tag{F.2}$$

From this we get the equations of motion

$$0 = \frac{\partial L_a}{\partial r^i_a} - \frac{\partial}{\partial t}\left(\frac{\partial L_a}{\partial v^i_a}\right) \tag{F.3}$$

$$\begin{aligned}
&= e_b e_b\frac{r^i_{ab}}{R^3_{ab}} + \frac{e_a e_b}{2c^2}\left(-\frac{r^i_{ab}}{R^3_{ab}}v^s_a v^s_b - \frac{3r^i_{ab}}{R^5_{ab}}v^s_a r^s_{ab}v^u_b r^u_{ab} + \frac{v^i_a}{R^3_{ab}}v^s_b r^s_{ab} + \frac{v^i_b}{R^3_{ab}}v^s_a r^s_{ab}\right)\\
&\quad - m_a\dot v^i_a - \frac{m_a}{2c^2}(\dot v^i_a v^2_a + 2v^i_a v^s_a \dot v^s_a) - \frac{e_a e_b}{2c^2 R_{ab}}\left(\dot v^i_b - v^i_b\frac{v^i_{ab}r^s_{ab}}{R^2_{ab}}\right)\\
&\quad - \frac{e_a e_b}{2c^2 R^3_{ab}}\left(v^i_{ab}v^s_b r^s_{ab} + r^i_{ab}\dot v^s_b r^s_{ab} + r^i_{ab}v^s_b v^s_{ab} - 3r^i_{ab}v^u_b r^u_{ab}\frac{v^s_{ab}r^s_{ab}}{R^2_{ab}}\right)
\end{aligned} \tag{F.4}$$

$$= -m\dot v^i_a + e_a e_b\frac{r^i_{ab}}{R^3_{ab}} + \frac{e_a e_b}{c^2}\left[-\frac{v^2_a}{2} - v^s_a v^s_b + \frac{v^2_b}{2}\right]\frac{r^i_{ab}}{R^3_{ab}}$$

$$+\frac{e_ae_b}{c^2}\left[-v_a^sv_a^i+v_a^sv_b^i\right]\frac{r_{ab}^s}{R_{ab}^3}-\frac{3e_ae_b}{2c^2}v_b^uv_b^s\frac{r_{ab}^ur_{ab}^sr_{ab}^i}{R_{ab}^5}+\frac{e_a^2e_b^2}{m_bc^2}\frac{r_{ab}^i}{R_{ab}^4} \tag{F.5}$$

Let us first compare the notation used in the various references,

$$\begin{array}{ccccccccc} Landau/Lifshitz & r_a^i & r_b^i & r_{ab}^i & R_{ab} & e_a & e_b & m_a & m_b \\ Wallace & \eta^i & \zeta^i & \beta_i & r & e_1 & e_2 & m_1 & m_2 \\ Gorbatenko & \xi^i & \eta^i & -R_i & R & Q & q & M & m \\ Jackson & r_1^i & r_2^i & r_{12}^i & R & q_1 & q_2 & m_1 & m_2 \end{array} \tag{F.6}$$

The Wallace[24] equations of motion (including radiation reaction term) are

$$\begin{aligned} m_1\ddot\eta^m+e_1e_2\frac{\partial}{\partial\eta^m}\left(\frac{1}{r}\right) &= e_1e_2\left[\left(\frac{1}{2}\dot\eta^s\dot\eta^s+\dot\eta^s\dot\zeta^s\right)\frac{\partial}{\partial\eta^m}\left(\frac{1}{r}\right)\right.\\ &\qquad\left.+\left(\dot\eta^s\dot\eta^m-\dot\eta^s\dot\zeta^m+\dot\zeta^s\dot\zeta^m\right)\frac{\partial}{\partial\eta^s}\left(\frac{1}{r}\right)-\frac{1}{2}\frac{\partial^3r}{\partial\eta^m\eta^r\eta^s}\dot\zeta^r\dot\zeta^s\right]\\ &\qquad-\frac{e_1^2e_2^2}{m_2}\frac{1}{r}\frac{\partial}{\partial\eta^m}\left(\frac{1}{r}\right)+\frac{2}{3}e_1(e_1\dddot\eta^m+e_2\dddot\zeta^m) \end{aligned} \tag{F.7}$$

Using

$$\frac{\partial}{\partial\eta^m}\left(\frac{1}{r}\right)=-\frac{\beta_m}{r^3},\quad \frac{\partial r}{\partial\eta^s}=\frac{1}{r}\beta_s,\quad \frac{\partial^2r}{\partial\eta^r\eta^s}=-\frac{\beta_r\beta_s}{r^3}+\frac{1}{r}\delta_{sr} \tag{F.8}$$

$$\frac{\partial^3r}{\partial\eta^m\eta^r\eta^s}=-\delta_{rm}\frac{\beta_s}{r^3}-\delta_{sm}\frac{\beta_r}{r^3}+\frac{3\beta_r\beta_s\beta_m}{r^5}-\frac{\beta_m}{r^3}\delta_{sr} \tag{F.9}$$

$$-\frac{1}{2}\frac{\partial^3r}{\partial\eta^m\eta^r\eta^s}\dot\zeta^r\dot\zeta^s=\frac{\dot\zeta^m\dot\zeta^s\beta_s}{r^3}-\frac{3\dot\zeta^r\dot\zeta^s\beta_r\beta_s\beta_m}{2r^5}+\frac{\beta_m\dot\zeta^s\dot\zeta^s}{2r^3} \tag{F.10}$$

we get

$$\begin{aligned} m_1\ddot\eta^m-e_1e_2\frac{\beta_m}{r^3} &= e_1e_2\left[-\left(\frac{1}{2}\dot\eta^s\dot\eta^s+\dot\eta^s\dot\zeta^s\right)\frac{\beta_m}{r^3}\right.\\ &\qquad\left.-\left(\dot\eta^s\dot\eta^m-\dot\eta^s\dot\zeta^m+\dot\zeta^s\dot\zeta^m\right)\frac{\beta_s}{r^3}-\frac{1}{2}\frac{\partial^3r}{\partial\eta^m\eta^r\eta^s}\dot\zeta^r\dot\zeta^s\right]\\ &\qquad+\frac{e_1^2e_2^2}{m_2}\frac{1}{r}\frac{\beta_m}{r^3}+\frac{2}{3}e_1(e_1\dddot\eta^m+e_2\dddot\zeta^m) \end{aligned} \tag{F.11}$$

$$\begin{aligned} &= e_1e_2\left[-\left(\frac{1}{2}\dot\eta^s\dot\eta^s+\dot\eta^s\dot\zeta^s-\frac{\dot\zeta^s\dot\zeta^s}{2}\right)\frac{\beta_m}{r^3}\right.\\ &\qquad\left.-\left(\dot\eta^s\dot\eta^m-\dot\eta^s\dot\zeta^m\right)\frac{\beta_s}{r^3}-\frac{3\dot\zeta^r\dot\zeta^s\beta_r\beta_s\beta_m}{2r^5}\right]\\ &\qquad+\frac{e_1^2e_2^2}{m_2}\frac{\beta_m}{r^4}+\frac{2}{3}e_1(e_1\dddot\eta^m+e_2\dddot\zeta^m) \end{aligned} \tag{F.12}$$

Translating this into the Landau/Lifshitz notation we see that it agrees with (F.5),

$$\begin{aligned} m_a\dot v^m-e_ae_b\frac{r_{ab}^m}{R_{ab}^3} &= e_ae_b\left[-\left(\frac{v_a^2}{2}+v_a^sv_b^s-\frac{v_b^2}{2}\right)\frac{r_{ab}^m}{R_{ab}^3}\right.\\ &\qquad\left.-\left(v_a^sv_a^m-v_a^sv_b^m\right)\frac{r_{ab}^s}{R_{ab}^3}-\frac{3v_b^rv_b^sr_{ab}^rr_{ab}^sr_{ab}^m}{2R_{ab}^5}\right]\\ &\qquad+\frac{e_a^2e_b^2}{m_b}\frac{r_{ab}^m}{R_{ab}^4}+\frac{2}{3}e_a(e_a\ddot v_a^m+e_b\ddot v_b^m). \end{aligned} \tag{F.13}$$

The Gorbatenko[61] equations of motion (including radiation reaction term) are

$$M\ddot{\xi}_k = -\frac{qQ}{R^3}R_k + qQ\left[\frac{(\dot{\xi}_l\dot{\eta}_l)}{R^3}R_k - \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\eta}_k + \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\xi}_k + \frac{(\dot{\xi}_l\dot{\xi}_l)}{2R^3}R_k \right.$$
$$\left. -\frac{\ddot{\eta}_k}{2R} - \frac{(R_l\ddot{\eta}_l)}{2R^3}R_k - \frac{3}{2}\frac{(R_l\dot{\eta}_l)^2}{R^5}R_k - \frac{(\dot{\eta}_l\dot{\eta}_l)}{2R^3}R_k\right] + \frac{2}{3}(Q\dddot{\xi}_k + q\dddot{\eta}_k)Q \qquad \text{(F.14)}$$

The Coulombian order equations for the η^k particle are the first two terms but with $\xi^k \to \eta^k$,$M \to m$, $Q \to q$, $q \to Q$, $R_k \to -R_k$. Using these equations we have

$$m\ddot{\eta}_k \approx \frac{qQ}{R^3}R_k \quad \Rightarrow mR_l\ddot{\eta}_l \approx \frac{qQ}{R} \quad \Rightarrow -\frac{(R_l\ddot{\eta}_l)}{2R^3}R_k \approx -\frac{qQ}{2m}\frac{R_k}{R^4} \qquad \text{(F.15)}$$

$$\Rightarrow -\frac{\ddot{\eta}_k}{2R} - \frac{(R_l\ddot{\eta}_l)}{2R^3}R_k \approx -\frac{qQR_k}{mR^4}. \qquad \text{(F.16)}$$

Substituting this last equation into (F.14) and assuming $1/(mR)$ is $\mathcal{O}(\lambda^1)$ gives

$$M\ddot{\xi}_k = -\frac{qQ}{R^3}R_k + qQ\left[\frac{(\dot{\xi}_l\dot{\eta}_l)}{R^3}R_k - \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\eta}_k + \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\xi}_k + \frac{(\dot{\xi}_l\dot{\xi}_l)}{2R^3}R_k \right.$$
$$\left. -\frac{qQR_k}{mR^4} - \frac{3}{2}\frac{(R_l\dot{\eta}_l)^2}{R^5}R_k - \frac{(\dot{\eta}_l\dot{\eta}_l)}{2R^3}R_k\right] + \frac{2}{3}(Q\dddot{\xi}_k + q\dddot{\eta}_k)Q \qquad \text{(F.17)}$$

Translating this into the Landau/Lifshifz notation we see that it agrees with (F.5),

$$m_a\dot{v}^k_a = \frac{e_be_a}{R^3_{ab}}r^k_{ab} + e_be_a\left[-\frac{v^l_av^l_b}{R^3_{ab}}r^k_{ab} + \frac{r^l_{ab}v^l_a}{R^3_{ab}}v^k_b - \frac{r^l_{ab}v^l_a}{R^3_{ab}}v^k_a - \frac{v^2_a}{2R^3_{ab}}r^k_{ab}\right.$$
$$\left. +\frac{e_be_ar^k_{ab}}{m_bR^4_{ab}} - \frac{3}{2}\frac{(r^l_{ab}v^l_b)^2}{R^5_{ab}}r^k_{ab} + \frac{v^2_b}{2R^3_{ab}}r^k_{ab}\right] + \frac{2}{3}(e_a\ddot{v}^k_a + e_b\ddot{v}^k_b)e_a. \qquad \text{(F.18)}$$

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